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An Integrated Behavioral Model of Land Use and Transport System: A Hyper-network Equilibrium Approach

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Abstract The interaction between the land use and transport in the urban context is a relevant issue in policy making. The connection between both systems arises since the former is causal of urban development while the latter is a consequence of it and significant contributor at the same time. One difficulty to unmask such interactions is to understand and determine the global system equilibrium, which is the matter of this paper. The households' decisions, from their residential location to their travel and route choices, are described as a process of interdependent discrete choices that reflect the long term equilibrium. Consumers are assumed to optimize their combined residence and transport options, which are represented as a set of paths in an extended network that includes the transport system together with fictitious additional links that represents land use and location market. At equilibrium no household is better off by choosing a different option for residential location or by choosing a different set of trips' destinations and routes. We study several alternative models starting from a simple case with fixed real estate supply and exogenous travel demand, to more complex situations with a real estate market, trip destination choices and variable trip frequencies. The equilibrium is characterized by an equivalent optimization problem which is strictly convex coercive and unconstrained. The optimality conditions for this optimization problem reproduce the transport equilibrium conditions as much as the land use equilibrium conditions. The approach provides a comprehensive characterization of the solution regarding existence and uniqueness, together with an algorithm to obtain the solution with well-defined convergence properties. The model is applicable to real size problems, with heterogeneous population and locations, as well as multiple trip purposes.

Keywords Integrated behavioral model · Land use · Transport system · Hyper-network · Equilibrium approach

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1 Introduction

One of the major complexities in modeling large urban areas for planning purposes is to properly represent the interactions between the transportation system and the spatial distribution of residential and non-residential activities. On the one hand, the spatial pattern of activities constitutes a major determinant of generation and attraction of trips from/to each zone, while travel times and costs are also a relevant input for location decisions through the measures of accessibility determined by the transportation system layout along with the demand conditions. Changes in land use and activities directly affect the transportation demand patterns, which in turn change the accessibility, and so on. This multilevel process defines a complex global equilibrium for the entire urban system, which has been explored to some extend but not yet solved by a methodology that reproduces the transport and land use equilibrium conditions simultaneously.

Several attempts to model the interaction between land use and transport (LU&T) in a partial equilibrium approach can be found in the specialized literature. Some land use models incorporate the transport dimension as an exogenous element for the location decisions, through a generalized measure of transportation costs which are considered to be fixed for the land use equilibrium mechanism. The generalized costs are normally obtained from a network equilibrium model which assumes in turn a fixed location pattern. These sequential equilibrium frameworks, known as bi-level or interactive models in the literature, do not solve the simultaneous land use-transport equilibrium, but instead they attempt to find the global equilibrium by means of iterative calculations of partial equilibrium of land use on the one hand and transport on the other. Apart from the computational effort involved in solving the iteration, the major drawback of these heuristic methods is that we can neither ensure the convergence of the procedure towards an equilibrium solution, nor the uniqueness of the limit point eventually attained. As far as we know, the existence and uniqueness of LU&T equilibrium remains so far as a largely open research topic.

In this paper we develop an integrated LU&T model based on a variational inequality formulation for the equilibrium. A strictly convex optimization problem is defined on an extended network, representing the discrete decisions not only taken at the transport system level but also at the land use system, both in the same graph. We ensure existence as well as uniqueness of the optimum under reasonable assumptions, and we propose a solution algorithm with guaranteed convergence towards the global LU&T equilibrium. The first order conditions of this problem reproduce the equilibrium conditions of two previously developed models, the "Random Bidding and Supply Model" (RB&SM) by Martínez and Henríquez (2007) for urban location and the "Markovian Traffic Equilibrium" proposed in Baillon and Cominetti (2006) for private urban transport networks. One important feature of these previous models is that all agents' decisions are modeled by discrete choice models, which provide the structural behavioral model of agents and defines the conditions to obtain a global equilibrium. It is worth noting however that we only allow negative externalities in the LU&T model, i.e. all interactions between agents, consumers or producers, are considered to be nuisances. This limitation is crucial in our variational inequality approach. Nevertheless, positive externalities may be represented if they are lagged in one period, so that they only affect the equilibrium as exogenous conditions.

In the next section we review some relevant previous attempts to model the interaction between land use and transportation equilibria.

2 Background

The challenge to model the integrated land use market and the transportation system (LU&T) has been faced using different mathematical approaches that also differ on their underpinning economic process. According to Chang (2006), the models can be categorized in Spatial Interaction, Mathematical Programming, Random Utility and Bid-Rent models. However, the problem of formulating the LU&T equilibrium has been so far described and formulated by using some simplified models (in most cases heuristics), for which there is no analytical way to establish conditions of existence, uniqueness and convergence to an equilibrium. This lack of consistency in modeling the global equilibrium yields researches with open questions on how the system actually behaves.

The first step to find an integrated formulation of LU&T was devoted to understand the basic relations behind the two subsystems. The spatial interaction model proposed by Lowry (1964) and later generalized in Wilson (1970), introduces the concept of cost impedance between zones, explicitly represented by a cost function. Wilson (1970) postulated a model based upon the maximization of the system entropy where, in addition to include fixed costs between zones, the author introduces a relative measure of the zone attractiveness. The model is not really able to completely explain the relation between land use and transport, mainly since it considers constant transportation costs.

The most relevant land use model in the context of this paper is the RB&SM (Martínez and Henríquez 2007) which belongs to Alonso's "Bid-Rent" approach (1964). In this model real estate transactions are commanded by an auction mechanism under the rule that the property is assigned to the highest bidder. In this scheme, the resulting willingness-to-pay for each location describes the behavior of the decision makers, as proposed by Alonso (1964). The RB&SM model is an extension of the Random Bidding Model (RBM) previously developed by Martínez and Donoso (2001) embedded in the operational software called MUSSA. The main features of the RB&SM model is that all consumers (households and firms) make their choices from a discrete set of locations and dwelling types, while suppliers provide discrete real estate options subject to comply with land use regulations and available land space. Consumers maximize a random utility (or surplus) and producers maximize a random profit modeled by logit models. At equilibrium all consumers locate in some of the supplied options, subject to the exogenous condition that total supply in the city equals total household population. This condition yields equilibrium as a fixed-point problem which is then solved by an iterative algorithm.

Another relevant land use model within the spatial interaction approach, is the doubly constrained entropy model (Roy 2004), similar to the proposal by Wilson (1970) with the difference that in this case the location is determined by the agents' willingness-to-pay assumed to be known. From this model, the logit probabilities proposed by Ellickson (1981) can be obtained. Following Anas (1981), the equivalence between the entropy maximization approach and the multinomial logit allows to reformulate the RB&SM model as a maximization of an entropy function (without externalities).

The aforementioned land use models find the equilibrium considering the transport system and trip costs as exogenously fixed variables.

The trip assignment models determine the route to be followed by each trip once the mode and destination have been chosen (see Ortúzar and Willumsen 1994). The situation has been frequently described as a game where traffic is seen as some sort of steady state in which travelers have no incentive to deviate from their current decisions. Since traffic involves many small players, a common approach is to ignore individual travellers and use continuous variables to represent aggregate average flows. Congestion is then modelled by flow-dependent travel times and a flow pattern is called an equilibrium if all used routes are optimal for these times. These aggregate models, also known as non-atomic or population games were introduced by Wardrop (1952) in a deterministic setting of identical players with perfect information. The variability in travel times and user perceptions led Dial (1971) to look at route selection in terms of random utility theory, and then a corresponding concept of *stochastic user equilibrium* (SUE) was investigated by Daganzo and Sheffi (1977). Wardrop equilibrium were characterized by Beckmann et al. (1956) as solutions of an equivalent convex minimization problem (see also Daganzo (1982) and Fukushima (1984) for a formulation using convex duality), and a similar characterization for SUE was obtained by Fisk (1980). For an historic account of traffic equilibrium we refer to the book Ben-Akiva and Lerman (1985) and the survey by Florian and Hearn (1999). A recursive property of the Logit model allowed Akamatsu (1997) to restate SUE in the space of arc flows, and inspired the general Markovian traffic equilibrium (MTE) in Baillon and Cominetti (2006). Unlike the previous equilibrium models which are route-based, the Markovian equilibrium models a chain of decisions where at each node the user decides the next link to get in, pursuing the minimization of the expected travel time to reach a predefined destination, regardless of the assignment decisions taken before.

One way to integrate the land use and transportation equilibrium problems is to find an equivalent mathematical programming formulation. Chang and Mackett (2005) formulate a bi-level problem to integrate both levels. At the superior level, the location problem is faced under a bid-rent approach by computing the access (accessibility and attractiveness) of the zones. At the inferior level, the network decisions are made taking into account the land use decided at the higher level. This procedure, however, does not ensure the existence of equilibria. Another model of this type is the one proposed by Boyce and Mattsson (1999) in which the equilibrium at the transport network level as well as that of land use are solved through optimization problems. The formulation satisfies equilibrium conditions at the transport level, however, there are no supply-demand market equilibrium conditions attained at the location level. Nagurney and Dong (2002) integrate the location choices in the network assignment problem as a variational inequality, reproducing the Wardrop conditions. In this approach the transport network is extended by artificial links that represent households' location decisions. The limitation of this approach is, however, that the land use market does not attain equilibrium because location choices are merely based on transport costs, ignoring land rents or assuming them exogenous.

The goal of this paper is to fully integrate the transport and the land use equilibrium models, by finding an equivalent mathematical programming formulation which ensures the existence and uniqueness of global equilibrium. Here, global equilibrium is characterized by a bid auction land market with variable supply and a transport system with elastic demand. Moreover, an effective algorithm to solve the model is proposed, ensuring convergence to the global equilibrium for the entire urban system.

Before formulating the model, a glossary of the problem variables is presented:

Exogenous data

N	set of nodes (i) in the transport sub-network
C	set of nodes representing household types (h) searching for a place to be located
H_h	number of households of type h
I	set of nodes (i) where households can get a location ($I \subseteq N$)
S_i	number of real estate supply units at zone i
D	set of destinations nodes (i) for the trips ($D \subseteq N$)
N_h^d	number of trips with destination d generated by each household of type h
A	set of links in the transport sub-network (between nodes in N)
$\widetilde{A}$	set of links in the location sub-network (from nodes in C to nodes in I)
i_a, j_a	tail and head nodes of link a
A_i^+	set of links whose tail node is i
A_i^-	set of links whose head node is i
$s_a(\cdot)$	travel time function of link $a \in A$ depending on the traffic flow
$\varphi_i^d(\cdot)$	discrete choice expected travel time function at node i for destination d

Steady state variables

t_a	travel time of link a
r_i	rent of a real estate located at zone i
b_h	monetary utility index (bid) for agent type h
H_{hi}	number of households located at node i

3 Integrated land use—transport model

3.1 Base model (fixed supply)

In the Markovian equilibrium scheme (MTE) by Baillon and Cominetti (2006), the authors develop a stochastic traffic equilibrium model on a transportation network assuming a known trip distribution pattern. They search for the equilibrium by means of the minimization of an objective function defined on the transport network. Our base model extends MTE by considering a more complex network in which fictitious arcs and nodes are added in order to represent the agents' location decisions. The original objective function is modified to represent the LU&T integrated problem.

Figure 1 depicts the extended network $G(N,A)$. In this scheme, the households are originally located at fictitious nodes g and h; each node holds different households groups. Each household choose their own optimal path along the location–transport system (dotted lines), competition for the locations is represented by the set of land use arcs incoming into each node in I from different households types nodes in C. These arcs transmit the willingness to pay of the agents for real estate options. In the scheme, the nodes i and j in I represents the available locations for the households, but they also belong to the transport network from which the members of the household will start their trips. The trip generation process is described by constant trip frequencies N_h^d. Each trip chooses an optimal path through the transport network and finishes at its destination node d. Only one transportation mode (private) is considered in this basic model.

The cost assigned to the location arcs represents the willingness-to-pay of the households for the land use at the location node (head of the arc). The location is decided through a bid-rent mechanism in the location decisions. Under this approach, the willingness-to-pay function represents the agents' behavior in the location decisions. The following willingness-to-pay function is postulated

$$B_{hi}(b_h, t) = -b_h + z_{hi} - \sum_{d \in D} N_h^d \tau_i^d(t) \qquad (1)$$

Here, b_h is a monetary disutility index (bid) for type agent h and z_{hi} captures how a household of type h values the set of attributes of zone i, including neighborhood quality. The transport effect is captured by the third component, in which functions $\tau_i^d(t)$ represent the expected time for a trip from node i to node d, and are characterized as the unique solution of the system of nonlinear equations $\tau_i^d(t) = \varphi_i^d(t_a + \tau_{j_a}^d(t); a \in A_i^+)$. The functions φ_i^d are given and describe the discrete

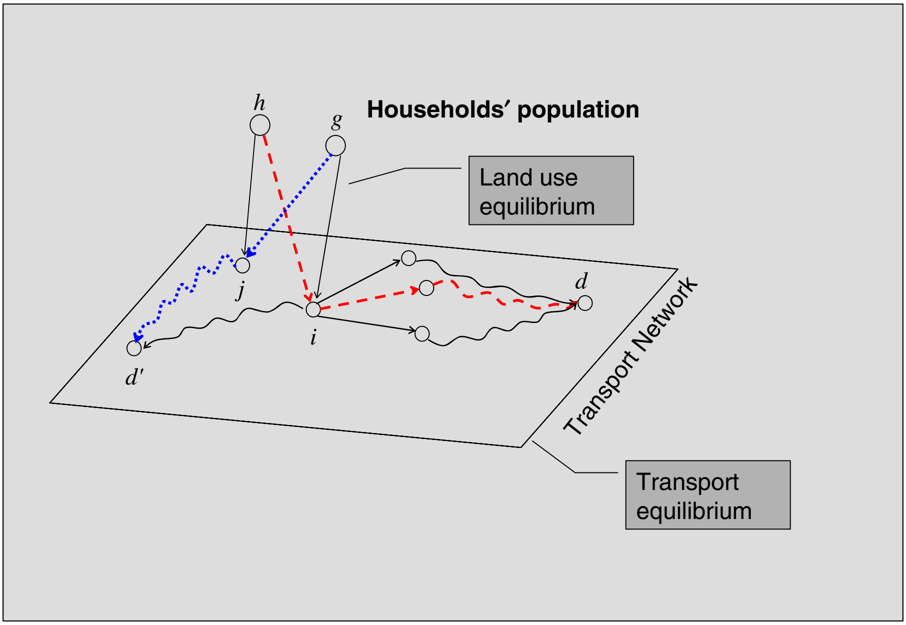

Fig. 1 Urban system: hyper-network representation

choice model used at node i for destination d. They belong to the class ε defined in Baillon and Cominetti (2006), *i.e.,* functions that can be expressed in the form $\varphi(x) = E(\min\{x_1 + e_1, \ldots, x_n + e_n\})$ where $e = (e_1, \ldots, e_n)$ is a random vector with continuous distribution and $E(e_i) = 0$. In the case of using the Gumbel distribution for these stochastic terms we get the usual log-sum expression $\varphi(x) = -\frac{1}{\beta}\log[\sum_i \exp(-\beta x_i)]$.

A technical point is worth mentioning. The form for the willingness-to-pay in Eq. (1) is derived from assuming: a quasi-linear subjacent utility function, i.e. linear utility in at least one of the consumption goods; an exogenous household income; and that the consumer chooses only one location for residence. It can be shown that under these assumptions the following relation between bids and utility levels holds: $b_h = (u_h/\eta_h) - y_h$, where y_h is the household income, u_h the utility level and η_h the marginal utility of income (Martínez and Henríquez 2007). Thus, in our model the first term in Eq. (1) shows the utility level reached by a household of type h, which has to be the same for all households of the same type under equilibrium conditions.

As was mentioned before, the use of an extended network to represent location choices was proposed previously by Nagurney and Dong (2002). However, the nature of what is represented by this extended network is radically different in our model. Contrarily to their definition where link costs are exogenous, here the willingness-to-pay of consumers presented in Eq. (1) contains endogenous information about the equilibrium, both in the transport market through travel times and in the land market through the utility term b_h. Thus, the global equilibrium represents the equilibrium conditions of two markets simultaneously, transport and land use, which are explicitly affecting the consumers' behavior.

We may now define the equivalent optimization problem which characterizes the combined LU&T equilibrium. The state variables (t, r, b) at equilibrium are found by solving the following optimization problem

$$\min_{t,b,r} \Phi(t, r, b) = \sum_{a \in A} \int_0^{t_a} s_a^{-1}(z)\,\mathrm{d}z + \sum_{h \in C} H_h b_h + \sum_{i \in I} S_i r_i + \frac{1}{\mu} \sum_{\substack{h \in C \\ i \in I}} e^{\mu(B_{hi}(b_h, t) - r_i)} \quad (2)$$

which combines the transport network problem, represented by the first term that depends on link travel time (t), and the land use problem represented by the next two sums, one across consumers depending only on utility (b), and the other one across location options that depends only on rents (r). The final sum combines all the variables establishing the link between the two markets.

It is worth commenting on how problem (2) departs from previous attempts to integrate land use and transport systems. Note first that Eq. (2) does not represent a classical network flow problem, as it was the case in the previous models, because in the land use sub-network the flows represent households seeking a location, while in the transport sub-networks, flows are expanded to represent individual trips. Then, note that the location equilibrium is not based on link congestion functions as in previous models, but on an auction mechanism which occurs at the location node level and matches the demand to the real estate supply.

In order to interpret problem (2), let us first concentrate in the last three terms of the objective function, that is, the terms related to the land market. Fixing the travel variables t yields a sub-problem in (r, b) that may be understood as the dual of the following doubly constrained maximum entropy problem (P):

$$(P) \quad \min_{H_{hi}} -\sum_{\substack{i\in I\\h\in C}} H_{hi}Z_{hi}(t) + \frac{1}{\mu}\sum_{\substack{i\in I\\h\in C}} H_{hi}[\ln(H_{hi}) - 1]$$
$$\text{s.a.} \quad \begin{array}{l} \sum_{i\in I} H_{hi} = H_h \text{ for all } \quad h\in C \\ \sum_{h\in C} H_{hi} = S_i \text{ for all } \quad i\in I \end{array}$$

where $Z_{hi}(t) = z_{hi} - \sum_{d\in D} N_h^d \tau_i^d(t)$. This problem represents the maximization of bids and describes Alonso's auction process for location. The Lagrange multipliers associated to the constraints in problem (P) are b_h and r_i respectively.

In this model, the number of households located at each zone i is bounded from above by the zone supply, which is assumed to be fixed. Similarly to the MTE model, the objective function in (2) is strictly convex and coercive, under the following assumptions:

(H_0):

- The functions $\varphi_i^d(\cdot)$ belong to the class ε (as defined above) with $\varphi_d^d \equiv 0$,
- The travel times $s_a(\cdot)$ are strictly increasing and continuous with $\lim_{x\to\infty} s_a(x) = \infty$,
- $t_a^0 = s_a(0) \geq 0$ and $\varphi_i^d(t^0) > 0$ for all $i \neq d$,

(H_1): b_1=0,
(H_2): $\sum_{i\in I} S_i = \sum_{h\in C} H_h$.

Condition (H_0) is the basic assumption in Baillon and Cominetti (2006) and is required to use their results. (H_1) is a normalization condition which is imposed in order to avoid the indetermination that results from the fact that the objective function in Eq. (2) is invariant to shifts in b and r, namely $\Phi(t, r, b) = \Phi(t, r + c, b - c)$ for every constant c. This lack of identification only reproduces the standard condition that static markets clear for relative values of prices (or rents in this model). Finally, (H_2) is necessary to ensure feasibility for problem (P) and states that total real state supply equals total demand. All these conditions combined ensure the existence and uniqueness of an optimal solution. The proof of this fact is somewhat technical so we refer to Briceño (2006).

We will next check that the first order optimality conditions for problem (2) characterize the equilibrium, not only for the bid-rent market as in the RB&SM, but also for the transport system replicating the traffic Markovian equilibrium MTE. Analytically, the first set of optimality condition gives

$$\forall h\in C \quad \frac{\partial\Phi}{\partial b_h}(t, b, r) = 0 \quad \Rightarrow \quad H_h = \sum_{i\in I} \exp\left(\mu(B_{hi}(b_h, t) - r_i)\right) \tag{3}$$

Defining $H_{hi} := \exp(\mu(B_{hi}(b_h, t) - r_i))$, interpreted as the total number oftype h households located at zone i, the following land-use equilibrium condition is fulfilled

$$H_h = \sum_{i\in I} H_{hi} \quad \forall h\in C, \tag{4}$$

which assures that all households in each type are allocated somewhere.

Moreover, by replacing the expression for $B_{hi}(b_h,t)$ given in Eq. (1) into Eq. (3) we obtain

$$\exp(-\mu b_h) = \frac{H_h}{\sum_{i\in I}\exp\left(\mu\left(z_{hi} - \sum_{d\in D} N_h^d \tau_i^d(t) - r_i\right)\right)} \tag{5}$$

which replaced back into the definition of H_{hi} yields

$$H_{hi} = H_h \times \frac{\exp\left(\mu(Z_{hi}(t) - r_i)\right)}{\sum_{j\in I}\exp\left(\mu\left(Z_{hj}(t) - r_j\right)\right)} = H_h \times \frac{\exp\left(\mu(B_{hi}(b_h,t) - r_i)\right)}{\sum_{j\in I}\exp\left(\mu\left(B_{hj}(b_h,t) - r_j\right)\right)} = H_h \times P_{i/h}. \tag{6}$$

In the latter equation $P_{i/h}$ represents the probability for a household of type h to prefer a real estate at zone i. Martínez (1992) obtains this probability, which is called "choice" probability, assuming that the households seek to maximize their surplus $\Delta_{hi}=B_{hi}(b_h,t)-r_i$ plus a stochastic term that distributes identical and independent (*iid*) Gumbel with dispersion parameter μ.

From Eq. (5) we can also deduce that for each $h\in C$ the following holds

$$b_h = \frac{1}{\mu}\ln\left(\sum_{i\in I}\exp\left(\mu(Z_{hi}(t) - r_i)\right)\right) - \frac{1}{\mu}\ln(H_h), \tag{7}$$

which reproduces the RB&SM market clearance equilibrium condition.

A second set of optimality conditions is obtained by setting to 0 the derivatives with respect to r_i, namely

$$\forall i \in I, \frac{\partial\Phi}{\partial r_i}(t,b,r) = S_i + \frac{1}{\mu}\sum_{h\in C}\exp\left(\mu(B_{hi}(b_h,t) - r_i)\right) \times (-\mu) = 0, \tag{8}$$

which yields

$$S_i = \sum_{h\in C}\exp(\mu(B_{hi}(b_h,t) - r_i)) = \sum_{h\in C} H_{hi}. \tag{9}$$

From this relation, we obtain

$$\exp(-\mu \times r_i) = \frac{S_i}{\sum_{i\in I}\exp(\mu \times B_{hi}(b_h,t))} \quad \forall i \in I \tag{10}$$

and then

$$H_{hi} = S_i \times \frac{\exp\left(\mu \times B_{hi}(b_h,t)\right)}{\sum_{g\in C}\exp\left(\mu \times B_{gi}\left(b_g,t\right)\right)} = S_i \times P_{h/i} \tag{11}$$

where $P_{h/i}$ is called the "bid" probability in RB&SM and represents the probability that the household type h is the highest bidder at location i competing with all bidders in C. This formula is derived in Martínez (1992) modeling the auction process by assuming bids distributed *iid* Gumbel with parameter μ (see also Ellickson 1981).

Moreover, from Eq. (10) we also obtain that for each $i \in I$

$$r_i = \frac{1}{\mu}\ln\left(\sum_{h\in C}\exp(\mu\times B_{hi}(b_h,t))\right) - \frac{1}{\mu}\ln(S_i), \tag{12}$$

recovering again rents at equilibrium obtained in the RB&SM model. This expression for r_i is interpreted as the maximum expected willingness-to-pay among the households asking for a place to be located.

The third set of optimality conditions provides the equilibrium on the transport network, obtained by differentiating with respect to the travel time variables

$$\forall a\in A,\quad \frac{\partial\Phi}{\partial t_a}(t,b,r) = s_a^{-1}(t_a) + \frac{1}{\mu}\sum_{\substack{h\in C\\ i\in I}}\exp(\mu(B_{hi}(b_h,t)-r_i))\times\mu\frac{\partial B_{hi}}{\partial t_a}(b_h,t),$$

with

$$\frac{\partial B_{hi}}{\partial t_a}(b_h,t) = -\sum_{d\in D}N_h^d\times\frac{\partial\tau_i^d}{\partial t_a}(t).$$

Then,

$$\frac{\partial\Phi}{\partial t_a}(t,b,r)=0 \quad\Rightarrow\quad s_a^{-1}(t_a) = \sum_{\substack{h\in C\\ i\in I}}\exp(\mu(B_{hi}(b_h,t)-r_1))\times\sum_{d\in D}N_h^d\times\frac{\partial\tau_i^d}{\partial t_a}(t)$$

$$=\sum_{d\in D}\sum_{i\in I}\left(\sum_{h\in C}H_{hi}N_h^d\right)\frac{\partial\tau_i^d}{\partial t_a}(t)$$

By defining $g_i^d := \sum_{h\in C} H_{hi}N_h^d$, which represents the total number of trips starting at zone i whose destination is d, we obtain

$$s_a^{-1}(t_a) = \sum_{d\in D}\sum_{i\in I}g_i^d\frac{\partial\tau_i^d}{\partial t_a}(t) = \sum_{d\in D}\left\langle g^d,\frac{\partial\tau^d}{\partial t_a}(t)\right\rangle = \sum_{d\in D}v_a^d = w_a, \tag{13}$$

reproducing the MTE equilibrium conditions for the flows in the transportation network, i.e., $t_a = s_a(w_a)$, where w_a is the total flow on link a.

The previous results show that the optimal solution of the optimization problem in Eq. (2) simultaneously satisfies the equilibrium conditions of both RB&SM and MTE.

3.2 The model with variable supply

In this section, we extend the model by relaxing the assumption of having a fixed supply, allowing supply levels to be obtained endogenously according to the features of the construction companies at each zone. We may think of housing units as a different type of flow that match at location nodes with households seeking houses. However these flows are not only of different magnitude compared with the transportation flows (as with households and trips above), but also they never enter into the transport network. In fact they flow through a sub-network orthogonal to the transport network, although they share the same spatial location nodes. This is schematically represented in Fig. 2 with additional links from real estate production companies, represented by nodes k and $k,'$ to the node in I; these links do not belong to the transport sub-network.

For this extension, the following optimization problem is formulated

$$\min_{t,r,b} \Phi(t,r,b) = \sum_{a \in A} \int_0^{t_a} s_a^{-1}(z)dz + \sum_{h \in C} H_h b_h + \sum_{k \in K} S_k \times \xi_k(r) + \frac{1}{\mu} \sum_{\substack{h \in C \\ i \in I}} e^{\mu(B_{hi}(b_h,t)-t_i)},$$
$$\text{with } \xi_k(r) = \frac{1}{\beta_k} \ln\left(\sum_{i \in I} \exp(\beta_k(r_i - c_{ik})) \right) \tag{14}$$

The model in Eq. (14) considers a set of construction firms ($k \in K$), with c_{ik} representing the cost for firm k to build at zone i. The total number of dwellings to be constructed by each firm is S_k. This new problem is also strictly convex and coercive under the assumptions (H_0), (H_1) and with (H_2) replaced by($\widetilde{H}_2$): $\sum_{k \in K} S_k = \sum_{h \in C} H_h$ which guarantees the existence of a unique global optimum for problem (13) (for the proof we refer again to Briceño (2006)).

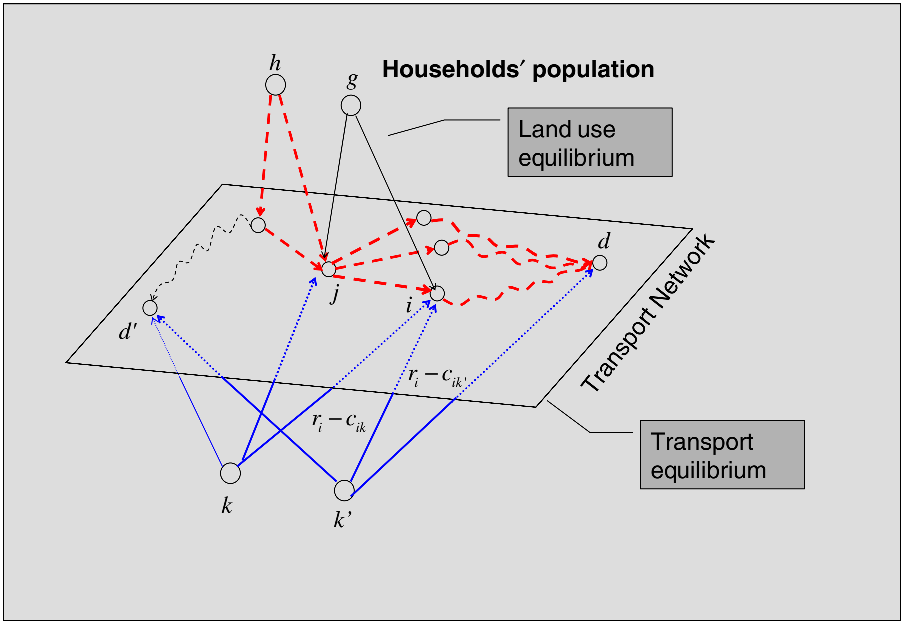

Fig. 2 Urban system hyper-network including variable supply

In this case, $\frac{\partial \Phi}{\partial b_h}(t,b,r)=0$ and $\frac{\partial \Phi}{\partial t_a}(t,b,r)=0$ replicate the same conditions as in the basic model. The novelty is in condition $\frac{\partial \Phi}{\partial r_i}(t,b,r)=0$. Analytically,

$$\frac{\partial \Phi}{\partial r_i}(t,b,r)=\sum_{k\in K} S_k\times\frac{\partial \xi_k}{\partial r_i}(r)+\frac{1}{\mu}\sum_{h\in C}\exp(\mu(B_{hi}(b_h,t)-r_i))\times(-\mu)$$

$$\frac{\partial \xi_k}{\partial r_i}(r)=\frac{\exp(\beta_k(r_i-c_{ik}))}{\sum\limits_{j\in I}\exp\left(\beta_k\left(r_j-c_{jk}\right)\right)}=:\pi_{i/k}, \tag{15}$$

Expression (15) shows the probability that firm k builds in zone i, which replicates the supply probability in the RB&SM model. Consequently,

$$\frac{\partial \Phi}{\partial r_i}(t,b,r)=0 \quad\Rightarrow\quad \sum_{k\in K} S_k\times\pi_{i/k}=\sum_{h\in C}\exp(\mu(B_{hi}(b_h,t)-r_i))=\sum_{h\in C} H_{hi} \tag{16}$$

which corresponds to the supply/demand equilibrium conditions in the RB&SM model (without scale economies). The left hand side of expression (16) quantifies the total number of dwellings built in zone i, while the right hand side is the total number of agents located at such a zone.

3.3 Including trips destination choices (fixed real estate supply)

In the models above the destination choice (trip distribution) is considered constant, with N_h^d providing the fixed number of trips made by the members of each household towards each destination. The main goal of this subsection is to add the distribution decision level, for which it is necessary to introduce the concept of trip purpose.

Typical trip purposes are work, study, shopping, etc. In order to make a trip of a specific purpose, the chosen destination has to be equipped to fulfill the purpose of the trip, imposing an additional constraint for such an assignment. In this model purposes are represented by extending the transport sub-network adding a new set of nodes reachable only from destinations that include the land use that fulfils the need of the specific trip purpose. Figure 3 graphically shows such a modeling scheme where multiple purposes are permitted.

In this representation of the urban system, the total number of trips generated by household and purpose is known. The destination choice depends on the minimum expected cost (time) from each origin to every other destination node linked to the purpose node. Then, for a specific purpose, the destination with the minimum expected time to get there is the one showing the highest probability of being chosen, among those destinations that comply with purpose's land use. The logit structure is kept for this additional decision level.

Let us denote P the set of purposes, and let us consider as known the number N_h^p of trips generated by each household of type h with purpose $p\in P$. The willingness-to-pay function is substituted by the following functional form

$$B_{hi}(b_h,t)=-b_h+z_{hi}-\sum_{p\in P} N_h^p\alpha_{hi}^p(t) \tag{17}$$

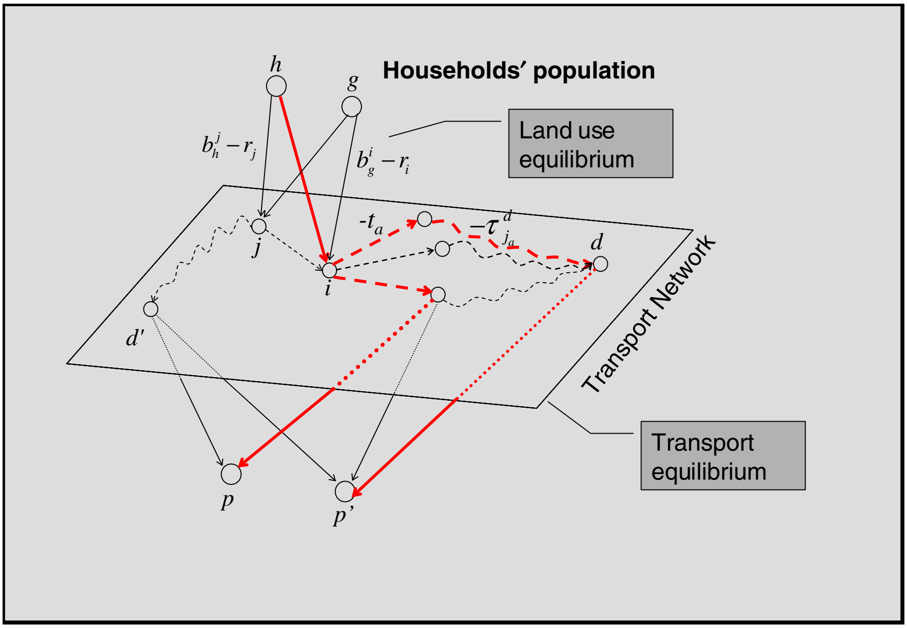

Fig. 3 Urban system hyper-network including trip purposes

where $\alpha_{hi}^{p}(t)$ represents the expected minimum cost to reach purpose p for a household of type h residing in zone i. In this scheme, costs have a random component which distributes *iid* Gumbel with scale parameter μ_{hi}^{p}. Analytically,

$$\alpha_{hi}^{p}(t) = -\frac{1}{\mu_{hi}^{p}} \ln\left(\sum_{d\in Q_p^-} \exp\left(-\mu_{hi}^{p} \times c_{hi}^{dp}(t)\right)\right), \tag{18}$$

where Q_p^- denotes the set of destination nodes from where purpose p can be reached through a single link, and $c_{hi}^{dp}(t) = \gamma_h^{dp} + \tau_i^d(t)$ represents the generalized cost of choosing d as destination including the minimum expected time $\tau_i^d(t)$ and the perceptions of other costs or benefits derived from attributes in location *d,* grouped in the term γ_h^{dp}. For instance, in the case of work trips this term can model the a priori importance of the facilities or employment at the work place. The term $\alpha_{hi}^{p}(t)$ takes into account only the generalized costs of those destinations that include the purpose p, which are defined through the extended network structure. The parameter μ_{hi}^{p} grows inversely proportional to the variance of the error terms in the generalized costs $c_{hi}^{dp}(t)$.

Under the assumptions of problem (2) with willingness-to-pay function as in Eq. (1), the same problem, but now with the modified willingness-to-pay function as in Eq. (17), has a unique solution at the global optimum. The same first order conditions of problem (2) apply, except again those associated to $\frac{\partial \Phi}{\partial t_a}(t,b,r) = 0$. Analytically,

$$\begin{aligned}\forall a \in A, \quad \frac{\partial \Phi}{\partial t_a}(t,b,r) &= s_a^{-1}(t_a) + \frac{1}{\mu}\sum_{\substack{h\in C\\ i\in I}} \exp(\mu(B_{hi}(b_h,t) - r_i)) \times \mu \frac{\partial B_{hi}}{\partial t_a}(b_h,t)\\ &= s_a^{-1}(t_a) + \sum_{\substack{h\in C\\ i\in I}} H_{hi} \times \frac{\partial B_{hi}}{\partial t_a}(b_h,t),\end{aligned}$$

where in this case

$$\frac{\partial B_{hi}}{\partial t_a}(b_h,t) = -\sum_{p\in P} N_h^p \times \frac{\partial \alpha_{hi}^p}{\partial t_a}(t) = -\sum_{p\in P}\sum_{d\in Q_p^-} N_h^p \times P_{d/hpi} \times \frac{\partial \tau_i^p}{\partial t_a}(t)$$

with $P_{d/hpi} = \frac{\exp\left(-\mu_{hi}^p \times c_{hi}^{dp}(t)\right)}{\sum\limits_{k\in Q_p^-} \exp\left(-\mu_{hi}^p \times c_{hi}^{kp}(t)\right)}$ which represents the logit probability of choosing destination d among those destination which serve purpose p on a trip purpose p by a consumer from household h located at i.

By defining $N_{hi}^{dp} := N_h^p \cdot P_{d/hpi}$, the number of trips with destination d, purpose p and indices (h,i,p), we obtain

$$\begin{aligned} \frac{\partial \Phi}{\partial t_a}(t,b,r) = 0 \quad \Rightarrow \quad s_a^{-1}(t_a) &= \sum_{\substack{h\in C\\ i\in I}} \sum_{p\in P} \sum_{d\in Q_p^-} H_{hi} N_{hi}^{dp} \frac{\partial \tau_i^d}{\partial t_a}(t) \\ &= \sum_{\substack{h\in C\\ i\in I}} \sum_{p\in P} \sum_{d\in Q_p^-} g_i^{dph} \frac{\partial \tau_i^d}{\partial t_a}(t) \\ &= \sum_{d\in D}\sum_{i\neq d}\left(\sum_{p\in A_d^+}\sum_{h\in C} g_i^{dph}\right)\frac{\partial \tau_i^d}{\partial t_a}(t), \end{aligned} \tag{19}$$

where $g_i^{dph} := H_{hi} N_{hi}^{dp}$ corresponds to the total number of trips generated by households of type h at zone i, with destination d and purpose p. Defining now $g_i^d = \sum\limits_{p\in A_d^+}\sum\limits_{h\in C} g_i^{dph}$ we obtain

$$s_a^{-1}(t_a) = \sum_{d\in D}\sum_{i\neq d} g_i^d \frac{\partial \tau_i^d}{\partial t_a}(t) = \sum_{d\in D}\left\langle g^d, \frac{\partial \tau^d}{\partial t_a}(t)\right\rangle = \sum_{d\in D} v_a^d = w_a.$$

from where we conclude that $t_a = s_a(w_a)$. Thus, once again the optimum satisfies the RB&SM and MTE equilibrium conditions simultaneously.

The previous model may be further extended in the following directions:

- The trip generation by household can be assumed dependent of the residential location zone i (independent of level of aggregation of purposes). That is, we may replace the exogenous constants N_h^p by N_{hi}^p. This modification does not alter the global equilibrium conditions LU&T.
- At the transport network level, the link selection process may be assumed to depend not only on the trip destination d but also on the household h that generates the trip as well as the trip purpose p. In order to do that, it is enough to consider functions $\tau_i^{dph}(t)$ instead of $\tau_i^d(t)$, which must now satisfy $\tau_i^{dph}(t) = \varphi_i^{dph}\left(t_a + \tau_{j_a}^{dph}(t); a \in A_i^+\right)$ with the family of functions φ_i^{dph} belonging to class ε. Under this new approach, we can get the first order conditions for the problem in Eq. (2) by replacing the willingness-to-pay function as in Eq. (17), and using the same definition of $\alpha_{hi}^p(t)$ with the difference that in this case $c_{hi}^{dp}(t) = \gamma_h^{dp} + \tau_i^{dph}(t)$. Analytically,

$$s_a^{-1}(t_a) = \sum_{h\in C}\sum_{p\in P}\sum_{d\in Q_p^-}\sum_{i\in I} H_{hi} N_h^p P_{d/hpi} \frac{\partial \tau_i^{dph}}{\partial t_a} = \sum_{h\in C}\sum_{p\in P}\sum_{d\in Q_p^-}\left\langle g^{dph}, \frac{\partial \tau^{dph}}{\partial t_a}\right\rangle \tag{20}$$

where $g_i^{dph} = H_{hi} N_h^p P_{d/hpi}$. By defining $v_a^{dph} = \left\langle g^{dph}, \frac{\partial \tau^{dph}}{\partial t_a} \right\rangle$ as in MTE, expression (20) corresponds to the total flow of trips traveling through link a. Thus,

$$s_a^{-1}(t_a) = \sum_{h \in C} \sum_{p \in P} \sum_{d \in Q_p^-} v_a^{dph} = w_a$$

where w_a is the total flow traversing link a as defined before, so that $s_a(w_a)=t_a$ preserving the LU&T global equilibrium conditions. In this approach the trips through links are disaggregated not only by purpose but also by both the type of household and the destination d of such a trip.

- If in the last framework the error terms of the link's travel time perception in the transport system distribute *iid* Gumbel, the functions $\tau_i^{dph}(t)$ will be log-sums of the form

$$\tau_i^{dph}(t) = -\frac{1}{\beta_i^{dph}} \ln\left(\sum_{a \in A_i^+} \exp\left(-\beta_i^{dph} \left(t_a + \tau_{j_a}^{dph}(t) \right) \right) \right).$$

The parameter β_i^{dph} grows inversely proportional to the variance of the stochastic travel time, so the larger this parameter, the more deterministic are decisions. One way to model such a phenomenon is to assume $\beta_i^{dph} = \beta^{ph} / L_i^d$, with L_i^d defined as the Euclidean distance between i and d.

This form implies that the ratio between the variability in travel time perceptions and distance for a given origin destination pair is constant. Thus, in longer trips the variability between alternative routes will be higher than that observed for shorter trips, which is equivalent to say that the variability in travel time perceptions grows proportionally with the distance of the trip. This seems reasonable, if we consider that one minute of time variability will induce less dispersion of trips as the travel time increases for different origin destination pairs. The underpinning logic of this argument comes from assuming a homogeneous structural variability through the network, which implies that as the trip length grows, the variability of travel time increases as it accumulates with the distance traveled. This technique also induces a considerable decrease in the number of parameters to be estimated, noting that the distance between origin and destination should be relatively easy to collect.

The link choice probabilities turn out to be:

$$P_{ij_a}^{dph} = \frac{\exp\left(-\beta_i^{dph} \left(t_a + \tau_{j_a}^{dph}(t) \right) \right)}{\sum_{b \in A_i^+} \exp\left(-\beta_i^{dph} \left((t_b + \tau_{j_b}^{dph}(t) \right) \right)} = \frac{\exp\left(-\beta^{ph} \left(t_a + \tau_{j_a}^{dph}(t) \right) / L_i^d \right)}{\sum_{b \in A_i^+} \exp\left(-\beta^{ph} \left(t_b + \tau_{j_b}^{dph}(t) \right) / L_i^d \right)}$$

which apparently implies that the probability of choosing link $a \in A_i^+$ grows with the expected "*speed*" for reaching d. However, this only apparent because the trip length for all alternatives is the same, so that what really matters for the decision is the minimum expected time.

3.4 Including trip generation

In the previous formulations real estate supply is fixed and, the number of trips generated by a household of type h is assumed to be known and exogenous. This assumption can be relaxed provided that a trip demand function by household can be estimated sensitive to changes in expected travel benefits or costs, so that the number of trips per household is elastic to such benefits or costs. Let us suppose we have trip demand functions $N_{hi}^{p}(\cdot)$ by purpose, zone and household type. In this generalized formulation the willingness-to-pay function becomes

$$B_{hi}(b_h, t) = z_{hi} - b_h - \sum_{p \in P} \int_0^{\alpha_{hi}^{p}(t)} N_h^p(x)dx. \tag{21}$$

Note that the distribution model without purpose selection corresponds to the particular case when $N_h^p(\cdot) \equiv N_h^p$ is constant.

In order to keep existence and uniqueness of the global optimum of problem (2), but with willingness-to-pay functions as in Eq. (21), it is enough to assume that the functions $N_h^p(\cdot)$ are decreasing and non negative, both assumptions quite reasonable in this context. As in the previous models, we will explore the first order conditions associated with $\frac{\partial \Phi}{\partial t_a}(t, b, r) = 0$ to visualize the equilibrium conditions under this new scenario. We obtain

$$\begin{aligned}\forall a \in A, \frac{\partial \Phi}{\partial t_a}(t, r, b) &= s_a^{-1}(t_a) + \frac{1}{\mu} \sum_{\substack{h \in C \\ i \in I}} \exp(\mu(B_{hi}(b_h, t) - r_i)) \times \mu \frac{\partial B_{hi}}{\partial t_a}(b_h, t) \\ &= s_a^{-1}(t_a) + \sum_{\substack{h \in C \\ i \in I}} H_{hi} \times \frac{\partial B_{hi}}{\partial t_a}(b_h, t)\end{aligned}$$

in which

$$\frac{\partial B_{hi}}{\partial t_a}(b_h, t) = -\sum_{p \in P} N_h^p\left(\alpha_{hi}^{p}(t)\right) \times \frac{\partial \alpha_{hi}^{p}}{\partial t_a}(t).$$

The term $N_h^p\left(\alpha_{hi}^{p}(t)\right)$ is the number of trips generated by a household of type h at zone i with purpose p, and depends on the minimum expected cost $\alpha_{hi}^{p}(t)$ required to accomplish the trip under such features. The transport network equilibrium conditions can also be obtained by following a similar procedure, ensuring the LU&T equilibrium as a whole.

4 Solution algorithm

In this section, we propose a solution algorithm to find the optimal solution of problem (2) with fixed real estate supply but with willingness-to-pay function including trip distribution and trip generation as in (17). The problem may be written in the equivalent form

$$\min_{t \in \mathbb{R}^{|A|}} \widetilde{\Phi}(t) = \sum_{a \in A} \int_0^{t_a} s_a^{-1}(z)dz + \Gamma(Z(t)) \tag{22}$$

where

$$\Gamma(z) = \min_{b,r} \sum_{i \in I} S_i r_i + \sum_{h \in H} H_h b_h + \frac{1}{\mu} \sum_{h \in C} \sum_{i \in I} \exp(\mu(z_{hi} - b_h - r_i)) \tag{23}$$

and $Z_{hi}(t) = z_{hi} - \sum_{p \in P} N_h^p \alpha_{hi}^p(t)$.

As the function $\Omega_z(b,r) = \sum_{i \in I} S_i r_i + \sum_{h \in H} H_h b_h + \frac{1}{\mu} \sum_{h \in C} \sum_{i \in I} \exp(\mu(z_{hi} - b_h - r_i))$ is strictly convex and coercive under the assumptions b_1=0 and $\sum_{i \in I} S_i = \sum_{h \in C} H_h$, then for each z there exist unique vectors $b(z)$ and $r(z)$ which are the optimal solution of Eq. (23) so that

$$\Gamma(z) = \sum_{i \in I} S_i r_i(z) + \sum_{h \in H} H_h b_h(z) + \frac{1}{\mu} \sum_{h \in C} \sum_{i \in I} \exp(\mu(z_{hi} - b_h(z) - r_i(z)))$$

The optimal solution vector may also be characterized by the following equations

$$b_h(z) = \frac{1}{\mu} \ln\left(\frac{1}{H_h} \sum_{i \in I} S_i \times \frac{\exp(\mu z_{hi})}{\sum_{g \in C} \exp\left(\mu\left(z_{gi} - b_g(z)\right)\right)} \right) = F_h(b,z) \tag{24}$$

$$r_i(z) = \frac{1}{\mu} \ln\left(\frac{1}{S_i} \sum_{h \in C} \exp(\mu(z_{hi} - b_h(z))) \right) = G_i(b,z) \tag{25}$$

The objective function $\widetilde{\Phi}$ under the same assumptions (H_0), (H_1), (H_2) is strictly convex and coercive, which ensures the existence of a unique optimal vector t^* solution of Eq. (22). We propose a gradient-like solution method to solve the problem. To this end we need to compute the gradient of $\widetilde{\Phi}$. It follows that

$$\frac{\partial \widetilde{\Phi}}{\partial t_a}(t) = s_a^{-1}(t_a) - \widetilde{w}_a(t) \tag{26}$$

where

$$\widetilde{w}_a(t) = \sum_{p \in P} \sum_{d \in Q_p^-} \sum_{h \in C} \sum_{i \neq d} g_i^{dph}(t) \frac{\partial \tau_i^{dph}}{\partial t_a}(t) \tag{27}$$

$$g_i^{dph}(t) = H_{hi}(t) N_h^p P_{d/hpi}(t) \tag{28}$$

$$H_{hi}(t) = \exp(\mu(Z_{hi}(t) - b_h(Z(t)) - r_i(Z(t)))) \tag{29}$$

$$P_{d/hpi}(t) = \frac{\exp\left(-\mu_{hi}^p c_{hi}^{dp}(t)\right)}{\sum_{k \in Q_p^-} \exp\left(-\mu_{hi}^p c_{hi}^{kp}(t)\right)} \tag{30}$$

$$c_{hi}^{dp}(t) = \gamma_h^{dp} + \tau_i^{dph}(t) \tag{31}$$

Algorithmically, the computation of the aggregate flow vector $\widetilde{w} = \widetilde{w}(t)$ required in Eq. (26) can be done by the following procedure:

(a) Iterate $\tau_i^{dph,n+1} = \varphi_i^{dph}\left(t_a + \tau_{j_a}^{dph,n}(t); a \in A_i^+\right)$ in order to estimate $\tau_i^{dph} = \tau_i^{dph}(t)$
(b) Compute $c_{hi}^{dp}(t)$, $\alpha_{hi}^{p}(t)$ and $Z_{hi}(t)$
(c) Iterate $b_h^{k+1} = F_h\left(b_h^k, Z(t)\right)$ to estimate $b_h = b_h(t)$ and $r_i(t) = G_i(b(t), Z(t))$
(d) Calculate $H_{hi}(t)$ and $P_{d/hpi}(t)$ by Eqs. (29) and (30) in order to compute $g_i^{dph}(t)$ by Eq. (28)
(e) Compute flows $x^{dph} = \left[I - \left(P^{dph}\right)'\right]^{-1} g^{dph}$ and $v^{dph} = \left(Q^{dph}\right)' x^{dph}$ as in MTE
(f) Aggregate flows $\widetilde{w} = \sum_{p \in P} \sum_{d \in Q_p^-} \sum_{h \in C} v^{dph}$.

We remark that the convergence of the iteration in step (a) has been proved in Baillon and Cominetti (2006), while the convergence of the iteration in step (c) is shown in Macgill (1977).

We may now describe the algorithm MLE proposed to solve the equilibrium. The iteration can be seen as a method of successive averages (MSA) on the aggregate flows w. The basic step is of the form

$$(\text{MLE}) w^{k+1} = w^k + \lambda_k\left(\widetilde{w}^k - w^k\right)$$

where $\widetilde{w}^k = \widetilde{w}\left(t^k\right)$ with $t^k = s(w^k)$, and the step size is chosen as $\lambda_k = 1/k$.

We observe that (MLE) may be equivalently written as a variable metric gradient iteration. Indeed, denoting $\Psi(w) = \widetilde{\Phi}(s(w))$ it readily follows that

$$\frac{\partial \Psi}{\partial w_a}(w) = \frac{\partial \widetilde{\Phi}}{\partial t_a}(s(w)) s_a'(w_a) = [w_a - \widetilde{w}_a(s(w))] s_a'(w_a)$$

and therefore we may rewrite (MLE) in the form

$$(\text{MLE}) w^{k+1} = w^k - \lambda_k D\left(w^k\right)^{-1} \nabla \Psi\left(w^k\right)$$

where $D(w)$ is a diagonal matrix with entries $s_a'(w_a)$. The convergence of the latter iteration towards the unique minimum of $\Psi(\cdot)$ is guaranteed by Theorem 4 in Baillon and Cominetti (2006), provided that the arc travel time functions $s_a(\cdot)$ are of class C^2. Since minimizing $\Psi(\cdot)$ is clearly equivalent to problem (22), we conclude that (MLE) converges toward the global LU&T equilibrium.

5 Simulations

The proposed algorithm was tested for the well known test network of Sioux Falls city (LeBlanc et al. 1975) shown in Fig. 4, which comprises 24 nodes and 76 links. Although the real city exists, we only used the network for testing the model, adding fictitious information on population and trip rates. Thus our simulations are fictitious.

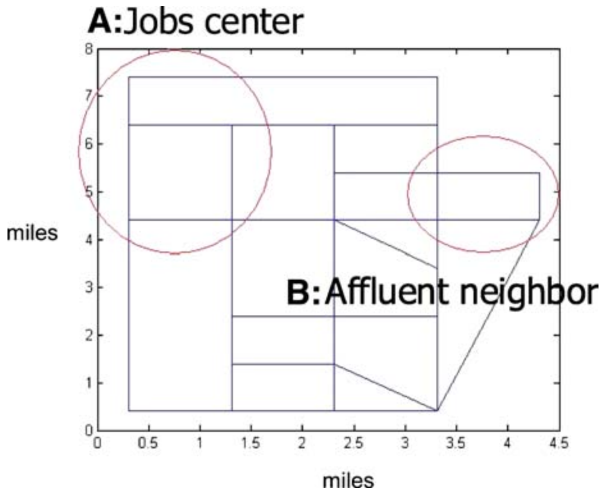

Fig. 4 Sioux Falls network and neighbors

We considered a fixed number of consumers and real estate supply units. The choice of an exit link at each node was modeled by a multinomial logit with a common scale parameter for all nodes, households, trip destinations and purposes. Thus,

$$\tau_i^{jph}(t) = -\frac{1}{\beta}\ln\left(\sum_{a\in A_i^+}\exp\left(-\beta\left(t_a + \tau_{j_a}^{jph}(t)\right)\right)\right).$$

The link travel time function follows the standard BPR form

$$s_a(w_a) = t_a^0\left(1 + b_a\left(\frac{w_a}{c_a}\right)^{p_a}\right)$$

which satisfies (H_0) and has no saturation capacity. Additionally, households were grouped into 5 income categories with 20 households each. We distinguish three categories of trip purposes (e.g. work, study and other) all of which are available only at five nodes (neighborhood A in Fig. 4). Total population is inelastic to equilibrium variables and 100 real estates are uniformly distributed among the 24 nodes of the network. We assume that, for location purposes, the household categories 1 and 2 (poor households) are attracted to the exogenous attributes of neighborhood A. The group of richer residents (categories 3, 4 and 5) is attracted to the affluent neighborhood B defined by four nodes. The remaining nodes are also available for location but all household categories are indifferent with respect to them.

In order to assess the performance of the equilibria under increasing population, we simulate a 10 years period with total population and real estate increasing each year by 20%, while trip generation growth rate is fixed to 10% per annum. These growth rates were chosen arbitrarily in order to make apparent the major impacts yield by the interaction between the transport and the land use systems, which we want to capture with the proposed formulation in order to validate the integrated optimization approach: this setting could be understood as the result of a dynamic fictitious simulation over a 10-year period. Namely, for years n=0...10 we set $H_h^n = H_h^0 \times (1.2)^n$, $S_i^n = S_i^0 \times (1.2)^n$ and $N_h^{p,n} = N_h^{p,0} \times (1.1)^n$. The initial trip generation rates $N_h^{p,0}$ were chosen so that the poor households generate 50% less trips than the rich ones.

The MLE algorithm was run for each subsequent year n=0, 1, …, 10. Iterations were stopped in each case as soon as we reached a precision of $\left\|\widetilde{w}^k - w^k\right\| \leq 10^{-9}$, which required on the average 220 iterations and 52.5 [s] of running time on a 3.2-GHz processor.

The following plots illustrate the interaction between population growth, location choices and transport congestion. Figure 5 shows how the average link times increase along the years due to higher congestion induced by the increments in population size and trip frequency.

As congestion increases the richer group is more rapidly affected because of their higher trip rates. This induces an increase of their willingness-to-pay to area A, where trip purposes can be performed, so that the rich group increasingly outbids the poor in that area, as shown in Fig. 6.

On the other hand, relative rents are affected by two economic effects: congestion and location externalities. In this model location externalities are neglected, hence in the simulation rents change only because of transport costs, which have negative effects on the average rents. Since in our model equilibrium is solved for relative rents only, Fig. 7 depicts the evolution of these state variables in the simulation. Average rents (denoted by "r" in the figure) decrease as congestion increases because consumers' willingness-to-pay is negatively affected for increase in trip times. In this figure the impact on rents is stronger on affluent neighbors because they lose rich population with higher willingness-to-pay, as shown in Fig. 6 above.

It remains to see to which extent the MLE algorithm performs better than a sequential bi-level approach (MTE&RBSM), which alternates iteratively between a full MTE transport equilibrium calculation and an RB&SM location equilibrium computation. Figure 8 compares the asymptotic convergence of both methods after

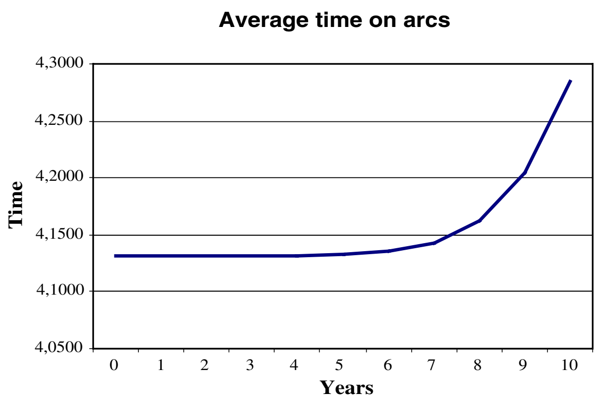

Fig. 5 Congestion caused by population and trip rates

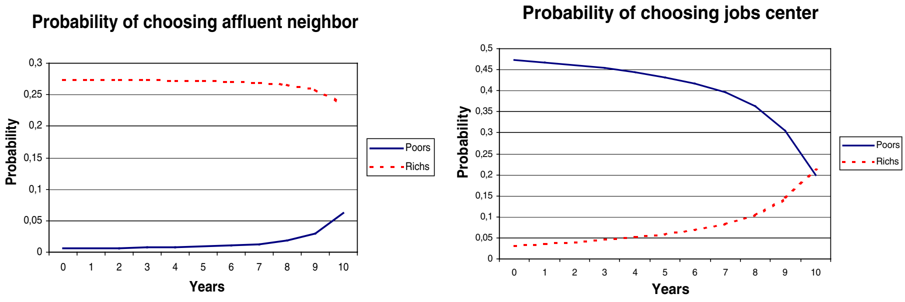

Fig. 6 Simulated residents share between poor and rich populations

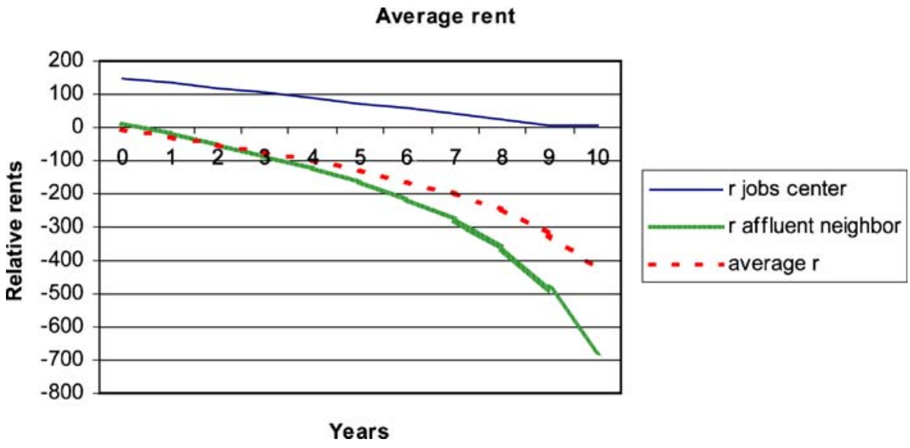

Fig. 7 Simulated rent changes

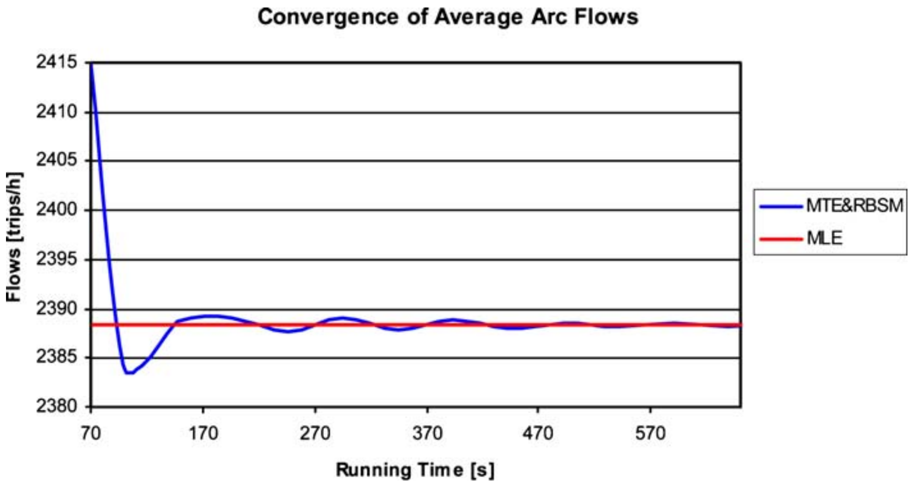

Fig. 8 Link flows convergence

70 [s] of running time. We observe that MLE has already attained a high precision before this time and its convergence is quite stable around 2388 trips per hour, which corresponds to the average arc flow at equilibrium. In contrast, the alternating MTE&RBSM method oscillates for a long time before stabilizing around the equilibrium. Furthermore we can not guarantee the convergence of the bi-level method.

6 Conclusions and further research

The models developed here allow the full integration of the land use and the transportation systems. The integrated model proposed is based on hyper-networks, i.e. an extension of the classical transport network to represent the land use market equilibrium. The modeling approach departs from the classical Beckman's approach used in previous models, of defining an equivalent optimization problem that reproduces Wardrop's traffic equilibrium conditions. That approach requires that flows are preserved in the network, while in our model links have different flows: households and housing units, on one sub-network, and households and trips in the other. Despite these different flows, they interact in the locations sites to attain equilibrium. Thus, the contribution of our model is to extend the original network approach and to define a new optimization problem that simultaneously reproduces the transport and land use equilibria; the original Beckman's integral is, nevertheless, incorporated in this problem. The main theoretical result is the proof of a unique solution for the LU&T system for several models: the basic land use and transport model and the extensions to variable real estate supply, destination choice and trips' frequencies.

In our analytical approach the network is extended by adding fictitious links and nodes, an idea previously used. However, the several extensions made required innovative modeling techniques, which makes that the network representation remains but the analytical methodology changes substantially compared to the classical transport network model. First, in the location problems the flows are households that once located generate trips, therefore flows are not preserved but expanded from households units to trip units. Second, modeling the real estate supply required defining a sub-network orthogonal to the transport sub-networks, because they share nodes but not links; in this sub-network the flows are housing units or households.

These extensions not only make possible to represent the cases presented, but they enlarge the classical network modeling approach, constrained to flat networks, to incorporate more complex cases, what we call the hyper-network approach. Thus, the hyper-network model of the urban system can be seen as a platform for modeling other dimensions of the urban system, as for example the information and the goods markets, as additional parallel or orthogonal layers in the hyper-network.

The model can be used under positive externalities by assuming that they are lagged in one or more periods, i.e. consumers make choices using information of the land use system that takes time to be acquired. In this dimension the model may be regarded as a partial equilibrium model and applied in a dynamic (time process) version.

One limitation of our model is the treatment of the transit system since only private transport modes have been considered. Public transport can be partially included by using a shortest path approach with congestion at bus stops in the spirit of De Cea and Fernández (1993) which is compatible with the proposed framework. However, a more complete treatment of the transit system including a fully congested strategy-based model such as the ones described in Cominetti and Correa (2001) or Cepeda, Cominetti and Florian (2006) remains as an open research question. Moreover, other public transport modes (such as light rail or metro) and also non motorized transport modes could be included into the modeling scheme, by adding other network layers in parallel to the private transport network with transfer links to model the interaction among different modes (see Baillon and Cominetti 2006).

Finally, the hyper-network approach can be used to specify dynamic urban processes on the hyper-network, including equilibrium stages along time on each submarket. This would allow considering delays in infrastructure development and lack of information on key variables of decision makers, like on expected future prices, in line with Martínez and Hurtubia (2006). The equilibrium problem so far developed provides the basic structure for such further extensions.

One further extension of our model is to explicitly include externalities, which are interactions among households and firms in their location choice and between residents and firms through traveling to pursue activities. Other interesting extensions to the framework could be: adding land regulations in the supply model; time and income constraints in consumers' behavior; allowing excess of either demand or supply, or both, in the land use equilibrium. In fact, allowing excess of demand/supply in the land use market convey to more realistic representation of the land market, and could be obtained by the dualisation of the doubly constrained maximum entropy problem (P) with inequalities in the constraints. This procedure requires both bounds to the rents and utility levels of the agents (represented by variables b_h, r_i), and therefore, the optimal conditions of the new formulation have to be rewritten. Moreover, the existence and uniqueness of the global optimum must be revised in the case of this more general formulation.

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