

An integrated behavioral model of the land-use and transport systems with network congestion and location externalities

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ABSTRACT

The agents' decisions, from their residential location to their members' trip choices through the network, are jointly analyzed as an integrated long term equilibrium in which the location, travel decisions, and route choices are represented by logit or entropy models. In this approach, consumers optimize their combined residence and transport options represented as paths in an extended network built by connecting the transport sub-network to a fictitious sub-network that represents land-use and transport demand options. We model a static *land-use and transport* equilibrium by considering road congestion and location externalities. The latter include trip destination choices based on land-use attractions, as well as endogenous neighborhood characteristics that determine residential choices and segregation phenomena. The model can deal with heterogeneous populations and locations as well as multiple trip purposes, though it assumes only private transport modes. In a previous paper we studied the case with road congestion externalities only, characterizing equilibria by a strictly convex and coercive unconstrained minimization problem. This characterization fails for more general externalities, so we restate the model as a fixed-point problem, establishing the existence of equilibria, providing sufficient conditions for its uniqueness and for the convergence of a fixed-point iteration. A small numerical example is used to illustrate the model.

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1. Introduction

One of the complexities in modeling big urban areas for planning purposes is to properly represent the interaction between the transportation system and the spatial distribution of residential and non-residential activities. This connection is twofold. On the one hand, the spatial pattern of activities represents a major determinant of generation and attraction of trips from/to each zone; while conversely, the transportation system is a relevant input for location decisions through the accessibility determined by the transport network and the demand conditions. Changes in land-use directly affect the transportation demand patterns, which in turn modify accessibility and location decisions, and so on. This process can be described either by the dynamic interaction between both sub-systems or by analyzing the global equilibrium of the combined *land-use and transport system* (LU&T).

Typically, land-use models include the effect of the transport system through generalized transportation costs which are taken as fixed for the location equilibrium mechanism. These costs are in turn obtained from network assignment models computed for a fixed location pattern. The system is simulated by iterative calculations of partial equilibrium of land-use on the one hand and transport on the other, leading to the so-called *interactive* or *bi-level* models. This approach does not

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provide a clear framework to analyze the existence and uniqueness of equilibria, while the iterations are not guaranteed to converge, with a further practical difficulty in terms of the high computational cost involved in the calculations.

An integrated LU&T model poses the challenge of solving simultaneously the internal conflicts within the transport and land-use sub-systems along with their interactions, namely

- *Location externalities within the land-use sub-system*: Conditional on the transport costs (accessibility), the location process depends on the built environment (buildings density) and the land-use that determines neighborhoods' amenities. Economists call these interactions location externalities, to emphasize that the location decision of one agent affects the utilities and decisions of others.
- *Congestion externalities within the transport sub-system*: Conditional on the location decisions taken as an exogenous factor, road congestion is an internal interaction among users in the transport sub-system where each route choice decision affects the travel time of all other users on the same route.
- *Trip attraction*: The choice of the trip destination is a decision that depends on both the activities located at each alternative spatial destination and the transportation costs.

These elements are mutually dependent and induce a price system that includes land rents and transport monetary costs. Moreover, consumer's time is a scarce resource that has to be rationally allocated by taking optimal decisions regarding spatial choices (residence and trip destinations) and transport choices (mode and route). An integrated equilibrium should consider these interactions and resource constraints, attaining all equilibrium conditions simultaneously. The major goal of the paper is to search for sufficient conditions to ensure the existence of equilibrium in the integrated system (land-use–transport) considering such interactions. Based on known approaches to model each stage, we formulate an integrated model that allows us to find such equilibrium including the added complexity caused by location externalities. Actually, to find a set of such conditions that can be mathematically verified is not a trivial problem in a context with externalities.

We build upon the integrated LU&T framework proposed in Briceño et al. (2008) using an extended network that expands the transport sub-network with additional nodes and arcs that represent land-use and transport demand options. In that model, the location, travel, and route choices are represented by Logit or entropy models, leading to a variational inequality for the simultaneous equilibrium of land-use, trip generation, distribution and network assignment, although only one transport mode – private car – is considered.¹ Equilibrium is represented through location and travel flows in the extended network, considering the effect of road congestion but ignoring the externalities associated with the location of agents (that can be households and firms). This simplified framework – adequate for short and medium term studies if one considers that land-use interactions evolve at a slower pace than transport – allows characterizing the global system equilibrium by an equivalent strictly convex optimization problem. This yields the existence and uniqueness of equilibria under weak assumptions, and leads to a globally convergent solution algorithm. The optimality conditions reproduce the equilibrium of two previous models: the *Random Bidding and Supply Model* (RB&SM) in Martínez and Henríquez (2007) for land-use equilibrium and the *Markovian Traffic Equilibrium* (MTE) in Baillon and Cominetti (2008) for private transport network assignment.

Our goal in the present work is to generalize Briceño et al.'s framework by adding location externalities to the model, while preserving the Logit structure for the individuals' choices. This generalized model recognizes that land-use externalities are an important feature when considering long term location decisions of agents. In the case of households, location choices of certain socioeconomic group members are affected by the socioeconomic characteristics of the neighbors resulting in segregation. In the case of firms, the interactions may either provide incentives to concentrate the activities at a given location in the form of *agglomeration economies*, or to disperse for competitive reasons. The extended network modeling approach is flexible enough to incorporate, in a coherent way, these multiple interactions in the urban system, including those associated with the movement of things and people (called transport), the location of activities (called land-use), as well as economic and social interactions. However, the model no longer has an equivalent optimization problem. Instead, we state the model as a multidimensional fixed point which integrates both sub-systems. Using this formulation we establish the existence of equilibria, we provide sufficient conditions for its uniqueness, and we propose a solution algorithm that converges towards equilibrium under appropriate assumptions.

The paper is structured as follows. In the next section, we briefly review some previous models dealing with the LU&T interactions, as well as the works which form the building blocks for our model. Section 3 describes the fixed-point formulation of the integrated LU&T equilibrium and presents the main results. In particular, Section 3.2 summarizes the additional properties that hold if one ignores location externalities, highlighting the equivalent optimization problem and how the corresponding optimality conditions reproduce the equilibrium in the land-use and transport markets. In Section 4 we report some small-sized simulation results to illustrate the model.

2. Brief literature overview

Several models dealing with the interaction between land-use and transportation are found in the literature. The survey by Chang (2006) categorizes the models in Spatial Interaction, Mathematical Programming, Random Utility and Bid-Rent

¹ Capacity constraints in public transport modes introduce significant additional complexities that require further research.

models. With the exception of the integrated LU&T model in Briceño et al. (2008) mentioned above, these models correspond to some form of interactive or bi-level approach often motivated by iterative computational schemes.

A bi-level approach was proposed in Chang and Mackett (2005). At the upper level, the location problem is faced under a bid-rent approach by computing the accessibility and attractiveness of the zones. At the lower level, the network decisions are made taking into account the access measures imposed by the upper level. This procedure, however, does not ensure the existence of equilibria. Another model of this type is the one proposed by Boyce and Mattsson (1999). They show that there exists a unique joint land-use and transport equilibrium for one household type and considering only one trip per household, a Walsarian-type of land-use equilibrium and a deterministic network assignment. The equilibrium is the optimum of the maximization of a strictly concave function defined in a large dimension space because of the route-based framework. In this model consumers' interactions are limited to the household's perception of the land-use density. In the present paper, we improve the treatment of the space dimension by considering an arc-based framework in the transport stochastic assignment that allows us to work with larger networks and a more realistic variability on travel cost perceptions. In addition, we consider different types of households and their number of generated trips and we also consider a more general way to model the externalities in the location market as well. Nagurney and Dong (2002) propose an integrated model expanding the transport network by including location choice links and formulating a location-assignment problem as a variational inequality that reproduces Wardrop's conditions. In this model, however, the land-use market does not attain equilibrium on land prices nor on externalities. Thus, as far as we know, the analysis of an integrated LU&T equilibrium with externalities remains open.

A first aspect for an integrated LU&T formulation is to properly model the trip structure. The spatial interaction model proposed by Lowry (1964) and later generalized by Wilson (1970), introduces impedances between zones through explicit cost functions and postulates a model based on the maximization of the system's entropy, introducing a relative measure of the zone attractiveness but considering fixed travel costs between zones. LU&T interactions may also be quantified by access measures derived from a microeconomic analysis as shown in Martínez (1995), and which allow to identify the relationship between user benefits from both the land-use and the transport systems (Martínez and Araya, 2000). These measures compute the transport benefits associated to origin and destination zones (accessibility and attractiveness) using a doubly constrained spatial interaction model.

A second feature to be considered is a mechanism to model the location decisions. For the present work, we adopt the RB&SM in Martínez and Henríquez (2007) which extends the *Random Bidding Model* by Martínez and Donoso (2001). In this framework, real estate transactions are commanded by an auction mechanism under the best bid rule, while the behavior of the decision makers is described by the willingness-to-pay for each location as proposed by Alonso (1965). A related land-use model within the spatial interaction approach is the doubly constrained entropy model proposed by Roy (2004), similar to the one by Wilson (1970) with the difference that in this case the location is determined by the agents' willingness-to-pay. This model yields the same logit probabilities proposed by Ellickson (1981) since the entropy maximization approach and the multinomial Logit are equivalent when parameters are estimated by a maximum likelihood method (Anas, 1981). Hence, the RB&SM may also be derived from an entropy maximization problem. These land-use models find the equilibrium considering the transport system and its interaction with the land-use market but, in all cases, the transport costs are exogenously determined.

The third component is a trip assignment mechanism to determine the route followed by each trip once mode and destination have been chosen (see Ortúzar and Willumsen, 1994). Traffic assignment has been traditionally modeled either by using Wardrop's conditions in a determinist setting or stochastic user equilibrium (Sheffi, 1985). In contrast with these route-based models, Baillon and Cominetti (2008) propose a Markovian equilibrium model for stochastic assignment in which route choice emerges from a chain of decisions where, at each intermediate node in his trip and regardless of the decisions taken before, the traveler uses a discrete choice model to decide the next arc to take in order to minimize his expected travel time-to-destination. As it follows from Akamatsu (1997), in a special case of Logit models with spatially uniform variance, this equilibrium is related (though not equivalent) to the stochastic user equilibrium.

3. Integrated land-use and transport model

3.1. General framework

The main goal of this work is to specify an integrated LU&T model and to prove existence of a static equilibrium for location, generation, distribution and assignment, considering externalities both in transport and location. In what follows, we focus the analysis on residential location and transportation problems; agents are households and individual travelers for the former and latter case, respectively. The modeling strategy is as follows. Consider an agent searching for a location and whose decision is affected by land-use externalities (which other agents are located at the zone and the quality of real estate infrastructure in the zone) as well as transport externalities imposed by the generation/distribution of trips and their assignment on the physical network. The transport equilibrium depends on the trip generation which depends in turn on the location of agents throughout the city. The presence of positive externalities does not allow characterizing the equilibrium by an optimization problem, so we use a fixed point approach by splitting the problem into: generation, distribution/assignment, and land-use equilibrium. Each stage captures the specific economic properties and internal interactions of one sub-system, while their coupling defines the global system equilibrium as a multidimensional fixed-point equation. Once the fixed point

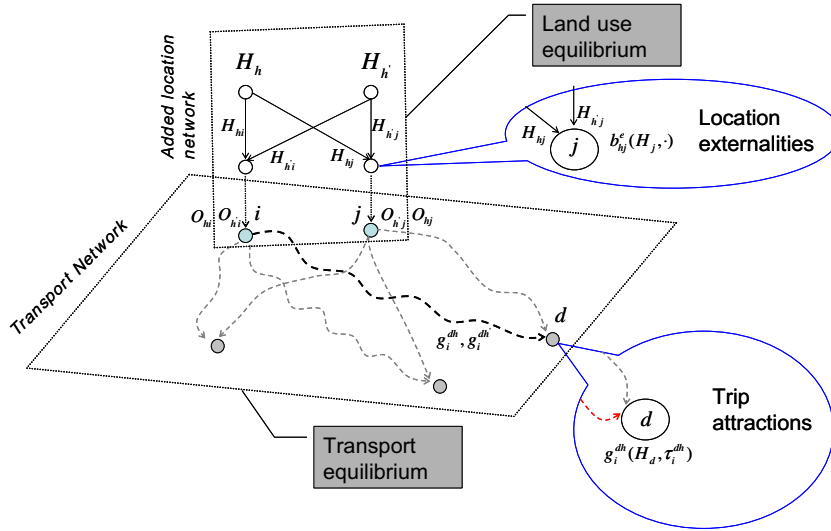


Fig. 1. Extended network representation of the urban system.

is properly stated to reflect the externalities across the whole system, the existence of equilibrium follows directly from Brower’s theorem. Fig. 1 below provides a graphical representation of the urban system viewed as an extended network, highlighting interactions among sub-systems.

A glossary of data and variables used to describe the model is as follows:

Exogenous data

- N set of nodes in the transport network
- A set of arcs in the transport network
- $D \subseteq N$ set of nodes representing destinations zones² $d \in D$
- $I \subseteq N$ set of nodes representing zones $i \in I$ where agents can get a location
- C set of fictitious nodes representing agent types $h \in C$ searching for a place to be located
- H_h total number of agents of type $h \in C$
- S_i total supply of real estate units at zone $i \in I$
- N_h^d number of trips generated from an agent of type h with destination d
- $s_a(\cdot)$ flow-dependent travel time function of arc $a \in A$
- $b_{hi}^e(\cdot)$ externality term of the willingness-to-pay function for agent type $h \in C$ dependant on zone $i \in I$
- $\gamma_d(\cdot)$ location externality function for trip attraction at destination $d \in D$
- $\varphi_i^{dh}(\cdot)$ discrete choice expected travel time from $i \in N$ to $d \in D$ for agent type $h \in C$

State variables

- $t = (t_a)$ with t_a the travel time on arc a
- $w = (w_a)$ with w_a the total flow on arc a
- $v = (v_a^{dh})$ with v_a^{dh} the flow of type h on arc a with destination d
- $g = (g_i^{dh})$ with g_i^{dh} the trips from i to d generated by agents of type h
- $H = (H_{hi})$ with H_{hi} the number of agents of type h located at zone i
- $O = (O_{hi})$ with O_{hi} the number of trips generated by agents of type h from zone i
- $b^u = (b_h^u)$ with b_h^u a monetary utility index for agents of type h
- $\alpha = (\alpha_{hi})$ with α_{hi} the Lagrange multipliers in the entropy model singly constrained on the trips generated by agents of type h at zone i (see details in Section 3.3)
- $r = (r_i)$ with r_i the real estate rent in zone i

Transport externalities are included in the maps $s_a: \mathbb{R} \rightarrow \mathbb{R}$ that give the arc travel times as strictly increasing continuous functions of the total arc flows w_a , namely

$$t_a = S_a(w_a) \tag{1}$$

² Physical space is represented by zones, and zones are represented by nodes in the extended network. Thus, hereafter, we refer to zones and nodes indistinctively.

Similarly, location externalities are incorporated through the following term of the *willingness-to-pay* maps

$$b_{hi}^e : \mathbb{R}^{3|C||I|} \times \mathbb{R}^{|A|} \rightarrow \mathbb{R}$$

that represent type h 's valuation of zone i 's attributes determined by the location variables H , the trip generation matrix O , the accessibility α obtained as the Lagrange multipliers of a singly constrained entropy model (see Section 3.3.2), and the travel times t . These maps, assumed of class C^2 , embed the interactions between transport and land-use. The location decisions are represented by H , which also provide the information required to calculate those attributes that define location externalities, while the interaction between land-use and transport is included as a functional dependency on the trips' generation matrix O and the travel times t that reflect the congestion on the transport network. The total willingness-to-pay b_{hi} , representing the *bids* of an agent of type h for a location at zone i , is composed by the externality and the utility terms:

$$b_{hi} = b_{hi}^e(H, O, \alpha, t) - b_h^u \quad (2)$$

where the variable $-b_h^u$ is the monetary disutility (less utility for higher bids) that endogenously adjusts the bids to ensure that all agents get located somewhere. Following Martínez and Henríquez (2007), this term is separable in the bid function as a direct consequence of assuming that the consumer's utility function underlying the willingness-to-pay function is quasi-linear.

The proposed fixed-point approach is as follows. We start from a given location matrix H in a simplex determined by total agents by type, total real estate supply, and non-negativity constraints. From this H , we compute the trip generation O by evaluating a simple affine function, after which the network assignment along with the trip distribution problems are solved simultaneously to obtain the transport network flows together with the arc travel times t . These are used in turn to evaluate the externality willingness-to-pay functions b_{hi}^e that enter the land-use equilibrium mechanism where utility b_h^u adjusts to ensure that all agents are located, producing a new location matrix \tilde{H} . Equilibrium is attained when this final location pattern \tilde{H} coincides with the initial one H , which yields a fixed-point equation that connects the sub-problems. In order to state the model, we begin in Section 3.2 by summarizing the case without location externalities, while in Section 3.3 each stage of the equilibrium procedure is presented highlighting their interactions. Section 3.3.4 analyzes the integrated LU&T equilibrium by putting all pieces together.

3.2. The model without location externalities

The Markovian traffic equilibrium (MTE) studied in Baillon and Cominetti (2008), is a stochastic traffic flow assignment model on a private transportation network for a given trip distribution $g_i^{dh} \geq 0$. In this setting, passengers travel to their destination by a recursive procedure in which an exit arc is randomly selected at every intermediate node in a trip, using a discrete choice model that seeks to minimize the expected time-to-destination. The expected travel times $\tau_i^{dh} = \tau_i^{dh}(t)$ from zone i to destination d for users of type h , are defined as the unique solution of the system of equations

$$\tau_i^{dh} = \varphi_i^{dh}(t_a + \tau_{j_a}^{dh} : a \in A_i^+) \quad \forall i \in N \quad (3)$$

where A_i^+ is the set of arcs whose tail node is i , j_a denotes the head node of arc a , and the maps $\varphi_i^{dh}(\cdot)$ characterize the discrete choice models used at that node. These maps belong to the class ξ of functions that can be expressed as an expected value of the form

$$\varphi(x) = \text{IE}(\min\{x_1 + \varepsilon_1, \dots, x_n + \varepsilon_n\})$$

where the $-\varepsilon_k$'s are random with continuous density and $\text{IE}(\varepsilon_k) = 0$. For iid Gumbel variables we obtain the usual log-sum expression $\varphi(x) = -\frac{1}{\beta} \ln[\sum_k \exp(-\beta x_k)]$.

The MTE is characterized as the optimal solution of the problem

$$\min_t \Phi(t) = \sum_{a \in A} \int_{t_a^0}^{t_a} s_a^{-1}(z) dz - \sum_{h \in C} \sum_{\substack{d \in D \\ i \neq d}} g_i^{dh} \tau_i^{dh}(t) \quad (4)$$

which turns out to be convex and coercive under the assumption

$$(\Pi_0) \begin{cases} \bullet \text{ the functions } \varphi_i^{dh}(\cdot) \text{ are of class } C^3 \text{ and belong to the class } \xi \text{ with } \varphi_d^{dh}(\cdot) \equiv 0, \\ \bullet s_a(\cdot) \text{ is strictly increasing and continuous with } \lim_{x \rightarrow \infty} s_a(x) = \infty, \\ \bullet t_a^0 = s_a(0) \geq 0 \text{ and } \varphi_i^{dh}(t^0) > 0 \text{ for all } i \neq d \end{cases}$$

The optimality conditions for (4) yield the equilibrium conditions

$$s_a^{-1}(t_a) = \sum_{d \in D: h \in C} v_a^{dh} = w_a \quad (5)$$

where w_a represents the total expected flow traversing the arc a since the expected flow of type h on arc a with destination d is given by (see Baillon and Cominetti, 2008)

$$v_a^{dh} = \sum_{i \in I} g_i^{dh} \frac{\partial \tau_i^{dh}}{\partial t_a} \tag{6}$$

Briceño et al. (2008) generalize the MTE model by considering an extended network in which fictitious arcs and nodes are added in order to represent the agents' location decisions and trip distribution. They modify the original MTE objective function to characterize the integrated LU&T equilibrium. In this formulation, the number of trips generated is constant for each agent type, while the destination in the transport network is implicitly decided by the optimal path. Location externalities are exogenous to the equilibrium process, so that a simpler expression for the externality willingness-to-pay is postulated as

$$b_{hi}^e(t) = z_{hi} - \sum_{d \in D} N_h^d \tau_i^{dh}(t)$$

where z_{hi} captures how an agent of type h values the attributes and location amenities of zone i , while the transport externalities appear in the second term with N_h^d the number of trips to destination d generated by an agent of type h , and $\tau_i^{dh}(t)$ the expected travel time from i to d as before. The integrated LU&T equilibrium is characterized by the problem

$$\min_{t,r,b} \Phi(t, r, b) = \sum_{a \in A} \int_{t_a^0}^{t_a} s_a^{-1}(z) dz + \sum_{h \in C} H_h b_h^u + \sum_{i \in I} S_i r_i + \frac{1}{\mu} \sum_{\substack{h \in C \\ i \in I}} \exp[\mu(b_{hi}^e(t) - b_h^u - r_i)] \tag{7}$$

which is strictly convex and coercive provided that (II_0) holds together with

$$(II_1) \sum_{i \in I} S_i = \sum_{h \in C} H_h,$$

$$(II_2) b_1^u = 0.$$

Condition (II_1) states that total real estate supply equals total demand, while (II_2) is a normalization to remove the indeterminateness that results from (II_1) since the objective function in (7) is invariant to shifts in b^u and r : $\Phi(t, r, b^u) = \Phi(t, r - c, b^u + c)$ for all $c \in \mathbb{R}$.³ The optimality conditions reproduce the MTE Eqs. (5) and (6) in the transport system with $g_i^{dh} = N_h^d H_{hi}$, as well as the land-use equilibrium reflected by the fact that all agents are located, namely $\sum_{i \in I} H_{hi} = H_h$. Here $H_{hi} = S_i P_{h/i}$ represents the total number of agents of type h located at zone i as described by a RB&SM in the bid-rent market, with the probability $P_{h/i}$ for an agent of type h to be the best bidder at zone i given by

$$P_{h/i} = \frac{\exp[\mu(b_{hi}^e(t) - b_h^u)]}{\sum_{g \in C} \exp[\mu(b_{gi}^e(t) - b_g^u)]} \tag{8}$$

3.3. The model with location externalities

In order to introduce location externalities to the integrated LU&T equilibrium, we restate the model as a fixed-point problem on the polytope of feasible locations defined by

$$K = \{H \in \mathbb{R}^{|C||I|} : H_{hi} \geq 0, \sum_{i \in I} H_{hi} = H_h \forall h \in C, \sum_{h \in C} H_{hi} = S_i \forall i \in I\} \tag{9}$$

We first describe the sub-models of generation, distribution/assignment, and location/bid; and then we analyze the integrated LU&T equilibrium as a fixed point of a composition map. As before, we ignore modal choice by assuming only one private transportation mode.

3.3.1. Trip generation

We denote $O = \Psi_1(H)$, where $\Psi_1 : K \rightarrow \mathbb{R}^{|C||I|}$ maps any given $H \in K$ to the trips generated by agents of type h from zone i , through the simple growth factor model⁴

$$O_{hi} = N_{hi} H_{hi} + \delta_{hi} \tag{10}$$

This model slightly deviates from the traditional fixed rate generation (where $N_{hi} > 0$ is assumed given) by adding a minimal given constant trip generation $\delta_{hi} > 0$, which ensures $O_{hi} \geq \delta$ for some strictly positive $\delta > 0$.

³ Here and afterwards, a sum $x + c$ of a vector x and a real c is to be interpreted as a component wise sum.

⁴ Here, for simplicity a linear model is considered. Nevertheless, this hypothesis can be strongly relaxed, as discussed later in this section. In general, the only condition required is a model like $O_{hi} = f_{hi}(H)$ with $f_{hi}(\cdot) \geq \delta > 0$ and f of class C^1 .

3.3.2. Trip distribution and network assignment

Trip distribution is described by a singly constrained maximum entropy model. Namely, considering as given the total number O_{hi} of trips of type h generated from zone i , the trips g_i^{dh} with destination d are obtained by solving the problem

$$(P) \quad \min_g \sum_{i \in I} \sum_{d \in D} \sum_{h \in C} c_i^{dh} g_i^{dh} + \sum_{h \in C} \frac{1}{\mu_h} \sum_{j \in I} \sum_{d \in D} g_j^{dh} (\ln g_j^{dh} - 1) \\ \text{s.t.} \\ \sum_{d \in D} g_i^{dh} = O_{hi}$$

where c_i^{dh} is the generalized cost of choosing d as destination, with a given Gumbel parameter $\mu_h > 0$. Equivalently, denoting α_{hi} the multipliers associated with the total trip constraints we may solve the dual problem

$$(D) \quad \min_{\alpha} \sum_{h \in C} \sum_{i \in I} O_{hi} \alpha_{hi} + \sum_{h \in C} \frac{1}{\mu_h} \sum_{i \in I} \sum_{d \in D} \exp[-\mu_h (c_i^{dh} + \alpha_{hi})].$$

We modify this dual formulation to postulate a joint distribution/network-assignment equilibrium described by the following optimization problem

$$\min_{\alpha, t} \Phi(\alpha, t) = \sum_{a \in A} \int_{t_a^0}^{t_a} s_a^{-1}(z) dz + \sum_{h \in C} \sum_{i \in I} O_{hi} \alpha_{hi} + \sum_{h \in C} \frac{1}{\mu_h} \sum_{i \in I} \sum_{d \in D} \exp[-\mu_h (c_i^{dh}(t, H) + \alpha_{hi})] \tag{11}$$

where the cost $c_i^{dh}(t, H) = \tau_i^{dh}(t) - \gamma_d(H)$ includes the expected travel time $\tau_i^{dh}(t)$ and the benefits derived from land-use attributes at location d grouped in the C^1 function $\gamma_d(H)$. The optimality conditions $\frac{\partial \Phi}{\partial t_a} = 0$ reproduce the MTE conditions (5) and (6) with

$$g_i^{dh} = O_{hi} \cdot P_{d/ih} \tag{12}$$

where

$$P_{d/ih} = \frac{\exp[-\mu_h (\tau_i^{dh}(t) - \gamma_d(H))]}{\sum_{k \in D} \exp[-\mu_h (\tau_i^{kh}(t) - \gamma_k(H))]}$$

is the probability of choosing zone d as destination, conditional on trip origin i and agent type h . Similarly, $\frac{\partial \Phi}{\partial \alpha_{hi}} = 0$ gives $O_{hi} = \sum_{d \in D} \exp[-\mu_h (c_i^{dh}(t, H) + \alpha_{hi})]$ which we rewrite as

$$\alpha_{hi} = \frac{1}{\mu_h} \ln \left(\frac{1}{O_{hi}} \sum_{d \in D} \exp[-\mu_h (\tau_i^{dh}(t) - \gamma_d(H))] \right) \tag{13}$$

In summary, problem (11) encompasses simultaneously the dual of the singly constrained distribution model and the MTE model. We denote $\Psi_2 : \mathbb{R}^{2|C||I|} \rightarrow \mathbb{R}^{|A|}$ the function that assigns to each couple (H, O) the optimal solution of (11), that is $(\alpha, t) = \Psi_2(H, O)$.

Lemma 1. Under assumption (II₀) the map Ψ_2 is well defined and of class C^1 .

Proof. For any given pair (O, H) problem (11) is strictly convex as well as coercive (a detailed proof of this Lemma is provided in Appendix) so there is a unique optimal solution and the map Ψ_2 is well defined. Moreover, the Implicit Function Theorem readily implies that this solution is smooth. \square

3.3.3. Best-bid auction and location mechanism

We assume that real estate transactions are commanded by an auction mechanism under the best bid rule. The bids for zone i are modeled as iid Gumbel variables, with given parameter $\theta_i > 0$, so that the probability for an agent of type h to set the highest bid and get located at zone i is

$$P_{h/i} = \frac{\exp[\theta_i b_{hi}]}{\sum_{g \in C} \exp[\theta_i b_{gi}]}$$

Thus, given a real estate supply S_i and denoting $b_{hi}^e = b_{hi}^e(H, O, \alpha, t)$ for brevity, the total number of agents of type h located at zone i is

$$\tilde{H}_{hi} = S_i \frac{\exp[\theta_i (b_{hi}^e - b_h^u)]}{\sum_{g \in C} \exp[\theta_i (b_{gi}^e - b_g^u)]} \geq 0. \tag{14}$$

Clearly $\sum_{h \in C} \tilde{H}_{hi} = S_i$ for all $i \in I$ so that all the supply is consumed. The bid adjustment mechanism determines the utility terms b_h^u in the agents' bids to make sure that at the same time all agents are located somewhere, $\sum_{i \in I} \tilde{H}_{hi} = H_h$ for all $h \in C$, so that $\tilde{H} \in K$. This yields the equations

$$\sum_{i \in I} S_i \frac{\exp[\theta_i (b_{hi}^e - b_h^u)]}{\sum_{g \in C} \exp[\theta_i (b_{gi}^e - b_g^u)]} = H_h \quad \forall h \in C \tag{15}$$

which are readily seen to be the optimality conditions for the convex program

$$\min_{b^u} \Gamma(b^u) = \sum_{h \in C} H_h b_h^u + \frac{1}{\theta_i} \sum_{i \in I} S_i \ln \left(\sum_{h \in C} \exp[\theta_i (b_{hi}^e - b_h^u)] \right). \quad (16)$$

From (Π_1) we get $\Gamma(b^u + c) = \Gamma(b^u)$ for constant c , so this problem has multiple solutions. However, the normalization (Π_2) further restricts the problem by $b_h^u = 0$ which becomes strictly convex and coercive (proof in Appendix) so that we recover a unique solution b^u . Recalling that b_{hi}^e is a function of (H, O, α, t) , we obtain that for any such tuple there is a unique vector b^u that adjusts the bids to ensure that all agents are located. We denote $\Psi_3: \mathbb{R}^{3|C||I|} \times \mathbb{R}^{|A|} \rightarrow K$ the map that gives the location pattern $\tilde{H} = \Psi_3(H, O, \alpha, t)$ derived from (14) for this particular b^u .

Lemma 2. Under (Π_1) and (Π_2) the map Ψ_3 is well defined and of class C^1 .

Proof. Once again this follows directly from the Implicit Function Theorem. \square

3.3.4. Fixed-point formulation of the integrated LU&T equilibrium

We proceed to formulate an integrated LU&T equilibrium model by connecting the three previous components. Starting from a given $H \in K$ we define the trip generation matrix $O = \Psi_1(H)$ by direct evaluation. The pair (H, O) then produces a vector of accessibility variables (α, t) through the map Ψ_2 (as well as the expected times to destination τ_i^{dh} and the arc flows v_a^{dh} and w_a). Finally, the tuple (H, O, α, t) determines a unique b and a corresponding location matrix \tilde{H} through the map Ψ_3 . Altogether, this composition gives \tilde{H} as a function of the initial location pattern H , namely, with $\Theta: K \rightarrow K$ given by $\Theta(H) = \Psi_3(H, \Psi_1(H), \Psi_2(H, \Psi_1(H)))$. An integrated LU&T equilibrium is then defined as a fixed point of the latter map, that is to say, a solution of the equation

$$H = \Theta(H). \quad (17)$$

Once a solution H is found, the remaining variables such as trip generation, distribution, network flows, and travel times, are readily computed by solving the optimization problems (11) and (17). Explicitly, the integrated LT&U equilibrium is defined by the equations:

$$O_{hi} = N_{hi} H_{hi} + \delta_{hi} \quad \forall i \in I, h \in C \quad (18)$$

$$t_a = s_a(w_a) \quad \forall a \in A \quad (19)$$

$$w_a = \sum_{d \in D, h \in C} v_a^{dh} \quad \forall a \in A \quad (20)$$

$$v_a^{dh} = \sum_{i \in I} g_i^{dh} \frac{\partial \tau_i^{dh}}{\partial t_a} \quad \forall a \in A, d \in D, h \in C \quad (21)$$

$$g_i^{dh} = \exp(-\mu_h(\tau_i^{dh}(t) - \gamma_d(H) - \alpha_{hi})) \quad \forall i \in I, d \in D, h \in C \quad (22)$$

$$\sum_{d \in D} g_i^{dh} = O_{hi} \quad \forall i \in I, h \in C \quad (23)$$

$$b_{hi} = b_{hi}^e(H, O, \alpha, t) - b_h^u \quad \forall i \in I, h \in C \quad (24)$$

$$H_{hi} = S_i \frac{\exp[\theta_i b_{hi}]}{\sum_{g \in C} \exp[\theta_i b_{gi}]} \quad \forall i \in I, h \in C \quad (25)$$

$$\sum_{i \in I} H_{hi} = H_h \quad \forall h \in C \quad (26)$$

where the time-to-destination functions $\tau_i^{dh}(t)$ are defined implicitly by the system

$$\tau_i^{dh} = \varphi_i^{dh}(t_a + \tau_a^{dh} : a \in A_i^+) \quad \forall i \in I, d \in D, h \in C \quad (27)$$

and the maps $s_a(\cdot)$, $\gamma_d(\cdot)$, $b_{hi}^e(\cdot)$, and $\varphi_i^{dh}(\cdot)$ are given functions that model, respectively, the flow dependent arc travel times, the attraction externalities at destinations, the willingness-to-pay functions for zones, and the discrete choice models for the traffic assignment.

Since K is a non-empty compact convex polytope and since the map $\Theta(\cdot)$ is continuous (as composition of continuous maps), Brouwer's Fixed-Point Theorem readily gives

Theorem 1. Assuming $(\Pi_0) - (\Pi_2)$ there is at least one integrated LU&T equilibrium.

Furthermore, the map is of class C^1 and since K is compact it turns out to be Lipschitz. We claim that when the θ_i 's are sufficiently small it is in fact a contraction.

Lemma 3. There exists $\theta_c > 0$ such the map $H \mapsto \Theta(H)$ is a contraction from K to itself as long as $\theta_i \in (0, \theta_c)$ for all $i \in I$.

Proof. Consider the maps $\kappa_{hi}(H) = b_{hi}^e(H, O, \alpha, t) - b_h^u$ with O, α, t and b^e expressed as functions of H using the maps Ψ_1 and Ψ_2 . Thus $\kappa_{hi}(H)$ is of class C^1 and using (14) we may express $\Theta_{hi}(H) = S_i \pi_{hi}(H)$ with

$$\pi_{hi}(H) = \frac{\exp[\theta_i \kappa_{hi}(H)]}{\sum_{g \in C} \exp[\theta_i \kappa_{gi}(H)]}$$

so we may compute the derivatives of the map $\Theta(\cdot)$ as

$$\frac{\partial \Theta_{hi}}{\partial H_{ij}} = \theta_i S_i \pi_{hi} \left[\frac{\partial \kappa_{hi}}{\partial H_{ij}} - \sum_{g \in C} \pi_{gi} \frac{\partial \kappa_{gi}}{\partial H_{ij}} \right]$$

then,

$$\frac{\partial \Theta_{hi}}{\partial H_{ij}} = \theta_i S_i \pi_{hi} (1 - \pi_{hi}) \left[\frac{\partial \kappa_{hi}}{\partial H_{ij}} - \sum_{\substack{g \in C \\ g \neq h}} \frac{\pi_{gi}}{1 - \pi_{hi}} \frac{\partial \kappa_{gi}}{\partial H_{ij}} \right]$$

By taking a uniform bound for the derivatives $\left| \frac{\partial \kappa_{gi}}{\partial H_{ij}} \right| \leq M$, and using that $\sum_{g \in C} \frac{\pi_{gi}}{1 - \pi_{hi}} = 1$, then since $\sum_{h \in C} \pi_{hi} = 1$ and $\max_{x \in [0,1]} x(1-x) = \frac{1}{4}$, we conclude that:

$$\left| \frac{\partial \Theta_{hi}}{\partial H_{ij}} \right| \leq 2M \theta_i S_i \pi_{hi} (1 - \pi_{hi}) \leq \frac{1}{2} M \theta_i S_i$$

It follows that $\Theta(\cdot)$ is a contraction provided the θ_i 's are small enough. \square

The interpretation of this result is straightforward: since the θ_i 's are inversely proportional to the variance of the Gumbel distributions of bids, the condition in Lemma 3 imposes a minimum dispersion of bids across consumers. On the contrary, if the agents behave increasingly deterministic, the equilibrium may be non-unique. In principle, once the modeling functions $s_a(\cdot)$, $\gamma_a(\cdot)$, $b_{hi}^e(\cdot)$, and $\varphi_i^{dh}(\cdot)$ are given, one may compute an estimate for M in order to get an explicit value for θ_c .

Notice that the Logit-type models used above assume finite and positive dispersion, which means that extreme cases, like the deterministic as well as the completely random cases are not included in the formulation.

Theorem 2. Assuming $(\Pi_0) - (\Pi_2)$ and $\theta_i \in (0, \theta_c)$ for all $i \in I$, there is a unique integrated LU&T equilibrium which can be computed by the convergent fixed-point iteration $H^{k+1} = \Theta(H^k)$.

Proof. This follows directly from Banach Fixed-Point Theorem. \square

Observation: This proof permits to have a complete vision of the stages involved in the implementation of the solution algorithm. Under all the previous assumptions, the algorithm works as follows. Given H^k , we compute O^k by means of Eq. (10). Then, the pair (α^k, t^k) is calculated by using a similar scheme as that proposed in Baillon and Cominetti (2008). After that, we construct the vector $(b_{hi}^e)^k$ to compute through the unique solution of problem (16) the associated vector H^k . Finally, H^{k+1} is computed by means of Eq. (25).

4. Simulations

The fixed-point iteration was tested on the network of Sioux Falls city shown in Fig. 2, which comprises 24 nodes and 76 arcs (LeBlanc et al., 1975). Although the real city exists, we considered fictitious data for population, trip rates, and real estate supply.

The procedure was implemented in MATLAB, using BPR arc travel time functions of the form $s_a(w_a) = t_a^0 [1 + b_a(w_a/c_a)^{p_a}]$. For the discrete choice in the MTE assignment we considered Logit models with scale parameter β independent of the node, agent type, and destination, so that the expected time-to-destinations were obtained by solving

$$\tau_i^{dh} = -\frac{1}{\beta} \ln \left(\sum_{a \in A_i^+} \exp[-\beta(t_a + \tau_{ja}^{dh})] \right)$$

Besides, we chose the following functional form for the willingness-to-pay functions

$$b_{hi}^e(H, O, \alpha, t) = z_{hi} + v_h \ln O_{hi} + \rho_h \alpha_{hi} + \Omega_h(H_i)$$

The parameter z_{hi} describes an exogenous value for the zone's features, while the term $v_h \ln O_{hi} + \rho_h \alpha_{hi}$ is the consumer's valuation of accessibility represented by the traveler's surplus obtained from the transport system. This surplus is obtained from the consumer's travel demand curve, balancing the benefits of activities and the trip costs reflected in the Lagrange multipliers α_{hi} of the singly constrained entropy model for trip distribution. The term $\Omega_h(H_i)$, taken as a linear function,

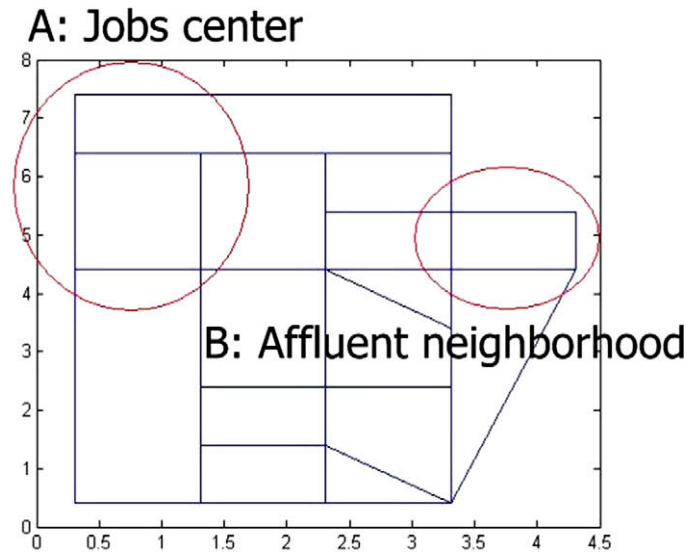


Fig. 2. Sioux Falls network and neighborhoods.

incorporates the location externalities by representing the like or dislike of agents of type h for other agents located at zone i . Finally, for simplicity, in these tests the functions $\gamma_d(H)$ were considered constants and the scale parameters of Eq. (25) are assumed $\theta_i = \theta \forall i$.

We considered five agent types divided in two groups: poor population (types 1–3) and rich population (types 4 and 5). Only one trip purpose was considered, with high attraction factors on five special nodes called neighborhood A or job's center, so that most trips are attracted towards this area. The constants z_{hi} were set to simulate a preference of the poor population for neighborhood A, while the rich population prefers the affluent neighborhood B represented by four neighboring nodes (see Fig. 2). Thus, if we ignore the congestion effects and location externalities, neighborhood A represents a poor residential area and an employment district, while neighborhood B is a rich residential area.

Simulations were run for seven scenarios indexed by $m = 1, \dots, 7$; varying the number of agents and real estate supply according to $H_i^m = 2^m H_i^0$ and $S_i^m = 2^m S_i^0$, but keeping fixed attraction factors. The distribution of households H_i^0 and real estate supply S_i^0 was taken homogeneous among agent types and location zones, respectively. Fig. 3 illustrates the equilibria obtained with and without location externalities: $\Omega(H) \neq 0$ and $\Omega(H) = 0$. In both situations, as the population grows the rich outbid the poor in neighborhood A which becomes more demanded as a consequence of increased network congestion. The outbid is less intense in the case with externalities, showing that segregation induces a higher preference of rich families for neighborhood B. Despite the higher complexity of the model with externalities, the running times in both cases are similar.

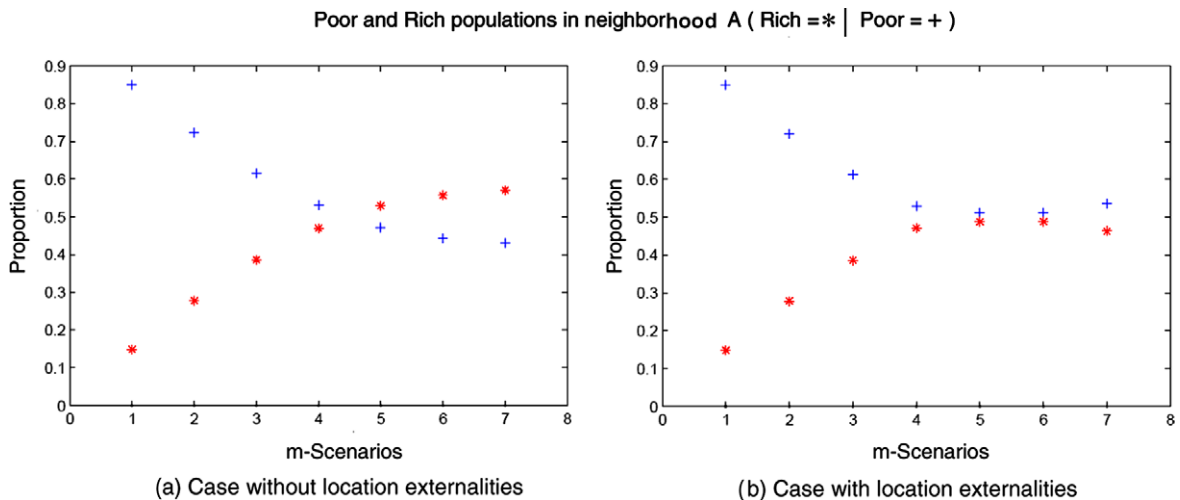


Fig. 3. Simulated resident's share between poor and rich populations with and without location externalities.

5. Conclusions and further research

We described a model to integrate the land-use and the transportation systems, including externalities in both sub-systems. The model extends the work of Briceño et al. (2008) where an integrated model was studied using an extended network representation of the joint land-use and transport equilibrium, which considered road congestion but ignored the externalities in the location process and trip attractions. The extended model was formulated as a fixed-point problem obtained by composition of smooth maps that describe trip generation, distribution/assignment, and bid/location. This fixed-point model was used to prove existence of equilibria, and to identify a mild condition on the dispersion of consumers' bids that guarantees uniqueness as well as convergence of a fixed-point iteration. This condition highlights the relevance of stochastic behavior, since as the behavior becomes closer to deterministic (low dispersion) the equilibria may no longer be unique. A strong assumption in the model is that bids are taken quasi-linear on utility levels.

The different stages of the transport and land-use models are treated by known modeling approaches, which provide interesting flexibilities in representing reality that are preserved in the integrated model. In that sense, we can highlight the following aspects: the wide scope of possible bid functions to be incorporated, the theoretical flexibility of the trip generation model (as detailed in Section 3.3.1), the different types of households considered, the general way to model the externalities in the location market, the space dimension considering an arc-based framework in the transport assignment that allows working with larger networks, among others. Besides, the integrated model represents an auction market and allows proving the convergence of the algorithm to the global equilibrium of the integrated system considering externalities in both location and traffic congestion, unlike the previous reported models that assume a Walrasian-type equilibrium and a heuristics to deal with similar integrated problems ST&US (bi-level) where the convergence to such an equilibrium is not ensured.

The proposed model considers only private transport modes in the transport system. Public transport can be partially included by using a shortest path approach in the spirit of De Cea and Fernández (1993). Also, public transport modes such as light rail or metro, as well as non-motorized transport modes, may be included in the model as suggested in Baillon and Cominetti (2008) by considering additional network layers in parallel to the private transport network with transfer arcs to model the interaction among different modes. However, a formal treatment of the congestion externalities in the transit system, including a fully congested strategy-based model such as the one described in Cominetti and Correa (2001) and Cepeda et al. (2006), remains open for further research.

The extended network framework may also be seen as a platform for modeling other dimensions of the urban system. Further developments may include the communications and the goods markets as additional layers in the extended network. Finally, the extended network approach can be used to study dynamic processes on the extended network, including equilibrium stages along time on each submarket in line with Martínez and Hurtubia (2006), by considering delays in slow moving variables such as infrastructure development, and the introduction of lack of information on key variables for decision makers such as expected future prices.

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Appendix A

Proof of Lemma 1. First, we must prove that Problem (11) has a unique solution. For that purpose, we will prove that the function $\Phi(\alpha, t)$ is coercive and strictly convex.

CONVEXITY. To prove strict convexity, we show that each component is strictly convex. In fact, the strict convexity of function $\psi(t) = \sum_a \int_0^{t_a} s_a^{-1}(z) dz$ comes from the fact that $s_a^{-1}(z)$ is assumed to be a strictly increasing function. In addition, the function $\zeta(t, \alpha) = \sum_{i \in I} \sum_{h \in C} O_{hi} \alpha_{hi} + \sum_{h \in C} \frac{1}{\mu_h} \sum_{d \in D} \exp(-\mu_h (c_i^{dh}(t) + \alpha_{hi}))$ is strictly convex with respect to α while the functions $c_i^{dh}(t)$ are concave with respect to t (see Baillon and Cominetti, 2008), from where $\zeta(t, \alpha)$ turns out to be strictly convex.

COERCIVITY. In this case, we have to prove that the recession function Φ^∞ of Φ is positive for all non-null direction, i.e.

$$\Phi^\infty(t, \alpha) = \lim_{\lambda \rightarrow +\infty} \frac{\Phi(\lambda t, \lambda \alpha)}{\lambda} > 0 \quad \forall (t, \alpha) \neq 0 \quad (\text{A})$$

First of all, let us analyze the case where $t \neq 0$. Hence,

$$\psi^\infty(t) = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \int_0^{\lambda t_a} s_a^{-1}(z) dz = +\infty$$

since $\lim_{w_a \rightarrow +\infty} s_a^{-1}(w_a) = +\infty$. Thus, it follows that in this case we have $\Phi^\infty(t, \alpha) = \infty$. Let us consider next the case where $t = 0$. By using calculus along with some properties of the recession function and composition, we have

$$\Phi^\infty(0, \alpha) = \sum_{i \in I} \sum_{h \in C} O_{hi} \alpha_{hi} + \lim_{\lambda \rightarrow +\infty} \sum_{h \in C} \frac{1}{\lambda \mu_h} \sum_{\substack{i \in I \\ d \in D}} \exp(-\lambda \mu_h ((\tau_i^{hd})^\infty(0) + \alpha_{hi}))$$

According to Baillon and Cominetti (2008), we have that $(\tau_i^{hd})^\infty(t) = \overline{\tau_i^{hd}}(t)$, where $\overline{\tau_i^{hd}}(t)$ is the minimum deterministic travel time between i and d when the cost of arc a is t_a . Therefore $(\tau_i^{hd})^\infty(0) = 0$ and then

$$\begin{aligned} \Phi^\infty(0, \alpha) &= \sum_{i \in I} \sum_{h \in C} O_{hi} \alpha_{hi} + \lim_{\lambda \rightarrow +\infty} \sum_{h \in C} \frac{1}{\lambda \mu_h} \sum_{\substack{i \in I \\ d \in D}} e^{-\lambda \mu_h \alpha_{hi}} \\ &= \sum_{i \in I} \sum_{h \in C} \underbrace{\left(O_{hi} \alpha_{hi} + |D| \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda \mu_h} e^{-\lambda \mu_h \alpha_{hi}} \right)}_{u_{hi}} \\ &= \sum_{(i,h) \in U^0} u_{hi} + \sum_{(i,h) \in U^+} u_{hi} + \sum_{(i,h) \in U^-} u_{hi} \end{aligned}$$

where $U^0 = \{(i, h); \alpha_{hi} = 0\}$, $U^+ = \{(i, h); \alpha_{hi} > 0\}$ and $U^- = \{(i, h); \alpha_{hi} < 0\}$.

We can see that, if $\alpha_{hi} \in U^0$ then $u_{hi} = 0$; if $\alpha_{hi} \in U^+$ then $u_{hi} = O_{hi} \alpha_{hi} > 0$ and if $\alpha_{hi} \in U^-$ then $u_{hi} = +\infty$. Hence, we have $\Phi^\infty(0, \alpha) > 0$ for all $\alpha \neq 0$ which completes the proof of coercivity. \square

Remark. Similar to the procedure used by Baillon and Cominetti (2008), we can prove that the functions $\tau_i^{hd}(t)$ are of class C^2 , by applying the implicit function theorem and taking into consideration that the $\tau_i^{hd}(t)$ functions are the solution of a fixed point-type equation. Then, we have that Φ is of class C^2 . By using again the implicit function theorem, we prove that Ψ_2 is of class C^1 which is the claim of Lemma 1 and the proof is complete.

Proof of strict convexity and coercivity of Problem 16. The reduced problem, after the normalization $b_1^u = 0$, is

$$\min_{b^u \in \mathbb{R}^{|\mathcal{C}|-1}} \Gamma(b^u) = \sum_{h \in C} H_h b_h^u + \sum_{i \in I} \frac{S_i}{\theta_i} \ln \left(\sum_{h \in C} \exp[\theta_i (b_{hi}^e - b_h^u)] \right)$$

CONVEXITY. To show that $\Gamma(b)$ is strictly convex, it is enough to prove that the function $f_i(y) = \frac{1}{\theta_i} \ln \left(\sum_{h \in C} \exp(\theta_i y_h) \right) = \frac{1}{\theta_i} \ln \left(\sum_{h \in C} \exp(\theta_i y_h) + 1 \right)$ is strictly convex. Let us define $z_h = \exp(\theta_i y_h)$ with $z \in \mathbb{R}^{|\mathcal{C}|-1}$. The Hessian associated with f is

$$H^f = \frac{1}{1^t z + 1} \text{diag}(z) - \frac{1}{(1^t z + 1)^2} \text{diag}(z^t z)$$

where 1^t is the row vector of ones with dimension $|\mathcal{C}| - 1$, and $\text{diag}(z)$ is the diagonal matrix with elements z_h . Now, let $0 \neq v \in \mathbb{R}^{|\mathcal{C}|-1}$. Then, we have

$$v^t H^f v = \frac{(\sum_h z_h v_h^2)(\sum_h z_h + 1) - (\sum_h z_h v_h)^2}{(\sum_h z_h + 1)^2} > \frac{(\sum_h z_h v_h^2)(\sum_h z_h) - (\sum_h z_h v_h)^2}{(\sum_h z_h + 1)^2} \geq 0$$

where the last inequality is the Cauchy–Schwartz inequality used with vectors $(v_h \sqrt{z_h})_h$ and $(\sqrt{z_h})_h$.

COERCIVITY. To prove that the function $\Gamma(b^u)$ is coercive, we write its associated recession function. Noting that the recession function of $\frac{1}{\theta_i} \ln \left(\sum_{h \in C} \exp[\theta_i (b_{hi}^e - b_h^u)] \right)$ and $g_i(b^u) = \frac{1}{\theta_i} \ln \left(\sum_{h \in C} \exp[-\theta_i b_h^u] \right)$ is the same (since the term b_{hi}^e is constant in this problem), we have

$$g_i^{\infty}(b^u) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda \theta_i} \ln \left(\sum_{h \in C} \exp[-\lambda \theta_i b_h^u] \right) = \min_h b_h^u = \overline{b^u}$$

with $\overline{b^u} \leq 0$, since $b_1^u = 0$. Then,

$$\Gamma^\infty(b^u) = \sum_h H_h b_h^u - \overline{b^u} \sum_i S_i \geq 0$$

where $\Gamma^\infty(b^u) = 0$ if and only if $\overline{b^u} = b_h^u = 0$ for all h . \square

For further details of these proofs and other associated developments, see Bravo (2007).

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