OPTIMAL CONTINUOUS PRICING WITH STRATEGIC CONSUMERS

Abstract. An important problem in electronic commerce is that of finding optimal pricing mechanisms to sell a single item when the number of buyers is random and they arrive over time. In this paper we combine ideas from auction theory and recent work on pricing with strategic consumers to derive the optimal continuous time pricing scheme in this situation. Under the assumption that buyers are split among those who have a high valuation and those having a low valuation for the item, we obtain the price path maximizing the seller's revenue. This amounts to conclude that, depending on the specific instance it is optimal to either use a fixed price strategy, or to use steep markdowns by the end of the selling season. To complement this optimality result we prove that under a large family of price functions there exists equilibrium for the buyers. Finally, we derive an approach to tackle the case in which buyers' valuation follow a general distribution. The approach is based on optimal control theory and is well suited for numerical computations.

1. Introduction

In many practical situations, particularly when selling items online, the precise number of potential consumers is unknown. Therefore, studying auction mechanisms with a random number of bidders has been an important question in economic theory since the work of McAfee and McMillan (1987). Significant amount of effort has been put in understanding these type of auctions under different assumptions (see e.g. Levin and Smith (1994); Levin and Ozdenoren (2004); Haviv and Milchtaich (2012)). However, these works assume a static situation in which the action takes place at one time (or in two rounds) and potential bidders are always present. Recent work in economic theory, including that of Board and Skrzypacz (2015) and Gershkov et al. (2014), consider a more general setting in which buyers arrive over time and fully strategize their decisions, however assume that the seller has full flexibility, and can design any type of mechanism, rather than just posted prices mechanisms more common in the pricing literature.

The issues of inter-temporal price discrimination and that of limited pricing flexibility have been central in the area of revenue management, which is concerned with price discrimination over time when selling perishable goods (see e.g. Talluri and van Ryzin (2005) for an in-depth treatment). Typically, in revenue management theory and practice firms use posted price mechanisms with a price that may vary over time. On the other hand, a frequent assumption in early revenue management work is that customers are not forward-looking. This assumption is usually violated since customers anticipate the pricing policy and incorporate such knowledge in their purchase decision, influencing as a consequence the firm's pricing decision. The evidence that consumers act strategically (Li et al., 2014) has opened a new line of research, analyzing conditions under which different pricing policies optimize the firm's profitability (Aviv and Pazgal, 2008; Cachon and Feldman, 2010; Caldentey et al., 2013; Correa et al., 2012; Caldentey and Vulcano, 2007; Elmaghraby et al., 2009; Jerath et al., 2010; Surasvadi and Vulcano, 2013; Osadchiy and Vulcano, 2010; Yin et al., 2009). In particular, pricing policies in which the price depends on the number of remaining items at the end of each period and fully preannounced pricing policies have been studied. A drawback of this literature is that it assumes that the price can only be changed at discrete time steps, usually considering two periods.

Key words and phrases. Revenue Management, Strategic Consumers, Optimal Pricing.

This work was supported by Iniciativa Científica Milenio through the Millennium Nucleus Information and Coordination in Networks RC130003. The first author was also supported by FONDECYT grant 11140360 and by Anillo ACT1106.

In this paper we address the problem of pricing an item over time when fully informed and forward-looking rational consumers arrive according to a random process, a common situation in electronic commerce. For instance *Mercado Minero* in Chile offers second hand mining machinery (mining being the largest industry in Chile) through a continuous pricing scheme[1]. In their business model they announce at time 0 the full price curve until the last day, say 180. Then, a consumer arriving at any time, may decide to buy upon arrival or wait until the price goes down enough. However, if a consumer decides to wait, she risks of not getting the item since another consumer may purchase it earlier. Other examples of companies using such announced pricing schemes include Lands End Overstocks and Dress for less. Also, companies such as TKTS in New York City and London announce the discount for the day of the show.

Our model differs from most of the revenue management literature in that the pricing can be adjusted continuously. In this sense our model is similar to that of Su (2007), although consumers' arrivals are modeled by a deterministic continuous flow. On the other hand, our work also differs from the recent literature in dynamic mechanism design, particularly the work of Board and Skrzypacz (2015) and Gershkov et al. (2014) in two key aspects. First, we only allow the seller to use posted prices rather than any type of mechanism. Second, and probably more important, is the fact that in these papers both, the buyers and the seller, discount the future at the same rate, which is key for a myersonian approach to go through. As argued, among others, by Pai and Vohra (2013) it is probably more realistic to assume that buyers are more impatient than the seller, therefore we follow the classic economic modeling of impatience and consider that buyers possess a temporal discount rate while the seller does not. It is worth noting that there are alternative models available in the literature, including those proposed by Su (2007), Mierendorff (2011), or Pai and Vohra (2013). In these models, buyers have an arrival time and either a time at which they leave the system or an explicit waiting cost.

Our results. We consider the situation in which a single item is to be sold over a time period. Two types of strategic buyers arrive over time: high value and low value consumers. Consumers arrive according to independent non-homogeneous poisson processes and upon arrival they decide when is most convenient for them to buy. To this end, they balance the price to be paid against the probability of getting the item, and discount the future at rate $\mu \geq 0$. The seller, who does not discount the future, decides an arbitrary price function over the time period in order to maximize his expected profit. Intuitively one may think that the optimal policy for the seller is to wait until the end of the time period and then perform a first price auction. We prove that this intuition is correct when $\mu = 0$, but fails in case $\mu > 0$. In this case, we explicitly find the optimal pricing policy for the seller and observe that offering steep discounts by the end of the period is optimal. Interestingly, we observe that the dynamic pricing mechanism obtains more revenue than that of an optimal auction. This additional revenue arises from exploiting the buyers' impatience.

To complement these results we show that for any fixed continuous price function, possibly with a discontinuity at the end of the season, a mixed strategy equilibrium for the buyers exists. We do this by constructing a symmetric equilibrium since the standard tools from fixed point theory do not apply to our setting. In this equilibrium, some buyers will buy upon arrival while others will use a mixed strategy over future times. Although not central to our paper this result may be of practical relevance since it allows to evaluate the expected revenue of wide array of price functions that may be needed if for instance there are regulations or practical constraints that impede the application of the optimal pricing.

Finally, we extend our approach to a more general setting in which buyers' valuations are arbitrary. Under an assumption on the buyers behavior, we formulate the seller's problem and apply optimal control theory to bring it down to solving a system of ordinary differential equations, which we are able to solve numerically.

[1] `http://www.mercadominero.cl/sitio/home.php?lang=1`

From a methodological perspective our work may be of interest since we depart from the myersonian approach and design a method based on optimal control formulations that explicitly consider the time. This seems to be crucial in order to tackle the different impatience degree of the buyers and the seller.

Assumptions. Let us briefly discuss the main assumptions we make in the subsequent analysis. First we assume that only two types of buyers valuation are present. Although this makes some of the analysis simpler, we believe that the main conclusions are not altered. Furthermore, this is a common simplifying assumption in the pricing literature (Caldentey and Vulcano, 2007; Su, 2007; Yin et al., 2009). In Section 4 we relax this two valuation assumption. Our second assumption on the consumer's side is that they arrive according to independent non-homogeneous poisson processes. This is more general than the homogeneous poisson arrivals imposed in part of the literature, and moreover there is strong empirical evidence that this is a good modeling assumption in electronic commerce (Russo et. al, 2010). The third assumption is that consumers discount the future at rate $\mu \geq 0$. Note that under this impatience measure, the buyers' behavior is straightforward: A higher value of μ make buyers more prone to buy earlier, increasing their chances of actually getting the item. From the seller we assume that he does not discount the future as it is reasonable that is more patient than buyers. Finally we assume that a single item is on sale. This is the case in many applications, and quite common in the economic theory literature and some of the revenue management work (see e.g. Caldentey et al. (2013)).

2. Model

Consider a risk neutral seller who wants to sell a single item over a season described by the time interval $[0, T]$. At time 0 the seller commits to a price function for the item $p(t)$, in case the item is available at time t. Consumers, who arrive according to a random process, take the price function as given and make purchasing decisions strategically. Namely, they decide to buy at a time (which is at least their arrival time) maximizing their expected utility, which balances the price to pay with the probability of actually obtaining the item. Naturally, we model the game as a stackelberg game (two stage dynamic game), where in the last stage strategic buyers seek to maximize profit under the imposed pricing, while in the first stage the seller selects a price function in order to maximize his or her own profit. We now describe the seller and buyers problems.

2.1. **The Seller.** The vendor's problem is to determine a price function $p(t)$. Since $p(t)$ will be public information and consumers are strategic, any choice of $p(t)$ induces a function $f(t)$, denoting the probability that the item is available at time t. Thus the seller, who is risk neutral and has the distributional knowledge of the buyers, needs to anticipate this function $f(t)$ and selects $p(t)$ to maximize his or her expected revenue.

2.2. **The Buyers.** There are two classes of buyers: *High value* consumers whose valuation for the item is V and *low value* consumers whose valuation for the item is $v < V$. We consider the buyers' discount rate to be $\mu \geq 0$. This discount rate models impatience on the buyers' side and also the fact that buyers may be risk averse. Both, the high value and low value buyers arrive to the store according to independent stochastic processes (counting process) with strictly positive interarrival times. We assume for simplicity that the arrival processes are non-homogeneous poisson processes of rates $\Lambda(t)$ for high value consumers and $\lambda(t)$ for low value buyers, though this assumption may be relaxed for some of our results. Let us then denote by $Q_k(t)$ (respectively $q_k(t)$) the probability that exactly k high value (respectively low value) buyers arrive in $[0, t]$. Note that, given that a buyer arrived at time $s < t$, these probabilities also represent the probability that k other buyers arrive in $[0, t]$. The informational assumption we make is that buyers know the arrival process and the fact that other buyers are strategic, but do not have information on how many other buyers have arrived at any point in time.

Given a price function p, a buyer arriving at time $t \in [0, T]$, with valuation $u \in \{v, V\}$ and who finds out that the item is still available, will buy at a time maximizing its own utility (provided that it is eventually nonnegative). Her utility at time $s \geq t$ is given by:

$$(u - p(s))e^{-\mu s}\mathbb{P}(\text{getting the item at time } s | \text{item is available at time } t).$$

Of course, for a buyer to get the item at time s, the item had to be available at time t. Thus, these events are included in one another, therefore the buyer will buy at $s \geq t$, solving

$$\begin{aligned}\max_{s \geq t} U(s) :&= \max_{s \geq t}(u - p(s))e^{-\mu s}\mathbb{P}(\text{getting the item at time } s) \\ &= \max_{s \geq t}(u - p(s))e^{-\mu s}\alpha(s)f(s).\end{aligned}$$

Recall that $f(s)$ denotes the probability that the item is available at time s, so that $\alpha(s)$ represents the probability of actually getting the item given that it is available in a random allocation model. In other words, if $r_k(s)$ represent the probability that there are k other buyers who chose to buy at time s, then $\alpha(s) = \sum_{k=0}^{\infty} r_k(s)\frac{1}{k+1}$. If the chances that two different buyers decide to buy in the exact same moment are zero, then this term is just $\alpha(s) = 1$. It is easy to observe that the latter happens for instance when the price function $p(t)$ is continuous, since in such a case a buyer may slightly anticipate the purchasing decision, paying only infinitesimally more but saving a significant amount in the term $\alpha(s)$.

2.3. **Equilibrium of the second stage.** Naturally, an equilibrium may be defined as a set of strategies for all potential buyers such that they cannot strictly improve their own profit by unilaterally deviating from the current situation. That is, the potential buyers $I = \{1, \ldots, n, \ldots\}$ that may arrive over $[0, T]$ must have a plan of action which is optimal given the plans of the other buyers. Even though our results apply to this general equilibrium concept, because players are ex-ante equal and in order to avoid excess of notation, we restrict our attention to symmetric equilibria.

Since the seller does not have incentives to lower the price below v the game actually occurs only among high value buyers. Consider thus a price function p such that $p(t) > v$ for all $t \in [0, T)$ and $p(T) = v$. We define a mixed strategy profile of the game as a family of distributions H_t over $[t, T]$, such that if a high value buyer arrives at time t, she buys at a random time chosen according to H_t. That is, we forget about defining a full plan of actions for every possible buyer and just index the strategies by the time when buyers arrive. We define equilibrium in terms of $f(t)$ accounting for the probability that the item is available at time t. Specifically, given a price function p such that $p(t) > v$ for all $t \in [0, T)$ and $p(T) = v$, we say that a probability function f *is an equilibrium for* p if there exist a family of distributions H_s over $[s, T]$, such that if a high value buyer arriving at time s buys at a random time chosen according to H_s, it holds that:

$$\begin{aligned} f(t) &= \text{prob. item is available at } t \text{ given buyers behaviour} && (1) \\ &= \sum_{k=0}^{\infty} Q_k(t) \int \prod_{i=1}^{k}(1 - H_{x_i}(t))dF_t(x_1, \ldots, x_k) \\ \text{supp}(H_t) &\subseteq \arg\max_{t \leq s \leq T}(V - p(s))e^{-\mu s}\alpha(s)f(s) && (2)\end{aligned}$$

Here $F_t(x_1, \ldots, x_k)$ denotes the conditional distribution of k high value buyers arrivals in $[0, t]$, subjet to one buyer has arrived at time 0.

This definition appears to be more natural and simpler: creating a full possible contingent of actions seems much harder to understand. Also, it is easy to see that defining for buyer $i \in I$ with arrival time t its contingent plan as H_t leads to a strategy in the classic bayesian setting. Therefore every equilibrium in our definition generates a Bayes-Nash equilibrium (a.e.). Also our definition only in terms of $f(t)$ is more robust in the sense that condition (2) could be imposed almost everywhere, so that if we change the distributions H_t for a negligible subset of $[0,T]$, $f(t)$ remains unaltered.

Note that considering mixed strategies is key since equilibrium in pure strategies may fail to exist. For the intuition behind this, suppose the price function p is continuous and that a buyer arriving at time t maximizes her utility buying at some time $s > t$ (i.e., the buyer chooses s deterministically). Then, since all buyers have equal valuation for the item, it is clear that every buyer arriving between t and s will also maximize her utility by buying at time s. But since there is a positive probability that someone arrives in $[t,s]$ and p is continuous, there exists a sufficient small ε such that $s-\varepsilon$ is a better option for the buyer arriving at time t, and therefore buying at s is not a best response for this buyer.

2.4. **Equilibrium of the first stage.** Given that the seller does not discount the future it is quite evident that his optimal strategy belongs to one of the following two families:

Constant Pricing: choose the price function $p(t) \equiv V$ (i.e., a no markdown strategy).

Markdown: choose a price function such that $p(t) > v$ for all $t \in [0,T)$ and $p(T) = v$.

Indeed, if $p(t) > v$ throughout the season, low valuation buyers will not buy, then setting the price to V is optimal. Otherwise, the price function will reach the value v at some point, but as the seller does not discount the future it is on his best interest to do this at the end since low valuation buyers will buy anyhow. In the former case, there is no strategic behavior: high value buyers buy upon arrival while low value buyers never buy. Thus, the expected revenue of the seller is easily computed as:

$$\pi(V) = V\mathbb{P}(\text{one or more high value buyers arrive}) = V(1 - Q_0(T)). \tag{3}$$

To express the expected revenue in the latter case observe that a price function induces an equilibrium, which determines an availability probability function $f(t)$. Thus, for $t < T$, $G(t) = 1 - f(t)$ can be seen as the distribution of the random variable expressing the time at which the item is sold considering only high value buyers, and for convenience we set $G(T) = 1 - Q_0(T)$. The latter holds since low value buyers only buy at time T. Thus, the vendor expected revenue under the markdown pricing scheme can be expressed as the integral of the price function with respect to the measure induced by G plus a term accounting for the probability of selling the item at price v when no high value buyer arrives:

$$\pi(p,G) := \int_0^T p(t)dG(t) + vQ_0(T)(1 - q_0(T)). \tag{4}$$

Note that in this expression the revenue corresponding to selling the item at price v is exactly v times the probability of selling the item at that price. Indeed, this probability is expressed as $Q_0(T)(1-q_0(T))$, in the second term of (4), plus the jump $G(T) - G(T^-)$ in the integral term (since $p(T) = v$). Overall this adds up to $1 - Q_0(T) - G(T^-) + Q_0(T) - Q_0(T)q_0(T) = 1 - Q_0(T)q_0(T) - G(T^-)$, i.e., the probability of selling the item minus the probability of not selling it before T, as desired.

In light of this observation, to determine the optimal pricing strategy, the seller needs to solve first the subproblem in which there are only buyers with valuation V and the price function has to be chosen among those satisfying: $p(t) > v$ for all $t \in [0,T)$ and set $p(T) = v$. In this subproblem buyers with valuation v only interfere as a threat to those of high valuation if they decide to postpone their purchase until time T. That is, the seller needs to find a price function maximizing $\tilde{\pi}(p,G) = \int_0^T p(t)dG(t)$. With this the seller may evaluate the integral in equation (4), then compare the quantities in (3) and (4) and select among these two the price function inducing the largest profit.

3. Optimal pricing

In this section we explicitly obtain the optimal price function, and its corresponding equilibrium for the seller. To this end we consider that only high value buyers arrive according to a non-homogeneous poisson process, and in Section 3.3 we plug this back into the sellers revenue to obtain the optimal profit. The space of functions over which the seller needs to make her selection, called $\mathcal{F}$, is the set of price functions $p\colon [0,T] \to [v,V]$ such that $p(T)=v$ and, for every $t \in [0,T)$, $p(t) > v$. Note that we do not impose any continuity or regularity of the price functions; we only require that an equilibrium for the buyers exists. By defining $\mathcal{D}$ as the set of non decreasing functions $G\colon [0,T] \to [0,1]$ such that $G(0)=0$, and setting

$$E = \{(p,G) \in \mathcal{F} \times \mathcal{D} : \, 1-G \text{ is an equilibium for } p\},$$

the vendor's subproblem may be written as:

$$\max_{(p,G)\in E} \tilde{\pi}(p,G) = \int_0^T p(t)dG(t).$$

To solve this problem we first compute an upper bound on the expected revenue of any price function in $\mathcal{F}$, and then find a particular pricing scheme whose expected revenue matches this upper bound. The tightness of this upper bound is based on two guesses that hold at the optimal price path: that high valuation buyers may buy as long as the item is available and that these buyers are indifferent between different purchase times. Thus our upper bounds essentially comes by lower bounding the utility of a buyer with what they would get if they buy at the end of the season. This translates into an upper bound on the seller's profit that we may match with a price function satisfying the guesses.

3.1. Upper bound on vendor's profit.

Recall that, given a function $p \in \mathcal{F}$, buyers choose a distribution function for deciding when they will buy so as to maximize their utility $U(s) = \alpha(s)(V-p(s))f(s)e^{-\mu s}$, where f is an equilibrium for p. So we let $(p,G)\in E$ and for convenience let $f = 1-G$. Thus, a buyer arriving at time t will get an expected utility of $u_t = \max_{s\geq t} U(s)$ and may buy at any time belonging to the set $S_t := \arg\max_{s\in[t,T]} U(s)$. Note that the value u_t is nonincreasing. In particular

$$u_t \geq Q_0(T)\alpha(T)e^{-\mu T}(V-v) \quad \text{ for all } t\in[0,T]. \tag{5}$$

Indeed the quantity on the right hand side is the expected utility the buyer would get if no other high value buyer arrives in $[0,T]$ and she gets the item in the random assignment at time T, so $\alpha(T) = \sum_{k=0}^{\infty} q_k(T)/(1+k)$. Let us define $S = \bigcup_{t\in[0,T]} S_t$, the set of times at which the item can be sold. Now, it follows from (5) that for all $t \in S$, i.e., a time at which there is interest in buying, we have that

$$U(t) = \alpha(t)f(t)e^{-\mu t}(V-p(t)) \geq Q_0(T)\alpha(T)e^{-\mu T}(V-v).$$

Using that $\alpha(t) \leq 1$, since it is a probability, we deduce

$$p(t) \leq V - \frac{Q_0(T)\alpha(T)}{f(t)}e^{-\mu(T-t)}(V-v) = V - \frac{Q_0(T)\alpha(T)}{1-G(t)}e^{-\mu(T-t)}(V-v) \quad \text{ for all } t\in S.$$

Hence, we have that:

$$
\begin{aligned}
\int_0^T p(t)dG(t) &= \int_S p(t)dG(t) \\
&\leq \int_S \left(V - \frac{Q_0(T)\alpha(T)}{1-G(t)} e^{-\mu(T-t)}(V-v) \right) dG(t) \\
&\leq \int_0^T \left(V - \frac{Q_0(T)\alpha(T)}{1-G(t)} e^{-\mu(T-t)}(V-v) \right) dG(t) \\
&= V \int_0^T dG(t) + Q_0(T)\alpha(T)(V-v)e^{-\mu T} \int_0^T e^{\mu t} \frac{-dG(t)}{1-G(t)}, \qquad (6)
\end{aligned}
$$

where the first equality follows since outside S the function $G(t)$ is constant, as no buyer buys outside S (the support of the measure induced by G is exactly S). The first inequality follows from the bound on $p(t)$ obtained earlier, while the second inequality is trivial. Hence, noting that $\int_0^T dG(t) = G(T) - G(0) = 1 - Q_0(T)$ for the first integral and using integration by parts for the second, we obtain that for all $(p, G) \in E$:

$$
\tilde{\pi}(p,G) \leq V(1-Q_0(T)) + Q_0(T)\alpha(T)(V-v)\left(\ln(Q_0(T)) - e^{-\mu T}\mu \int_0^T e^{\mu t} \ln(1-G(t))dt \right). \quad (7)
$$

3.2. **Matching upper bound.** Now we provide a pricing scheme, together with a very natural equilibrium for it, that attains the latter upper bound for the revenue of the vendor. We then conclude that the revenue the seller obtains under this price function is best possible for the subproblem in which only high value consumers arrive. Denote by

$$
p^*(t) = \begin{cases} V - \frac{Q_0(T)\alpha(T)}{Q_0(t)} e^{-\mu(T-t)}(V-v) & \text{for } t \in [0,T) \\ v & \text{for } t = T, \end{cases}
$$

and observe that, for every $t \in [0,T)$, $p^*(t) > v$ and $p^*(T) = v$. Note also that this function is discontinuous at time T. Under this pricing scheme, a buyer arriving at time t will seek to maximize

$$
\max_{s \in [t,T)} f(s)\alpha(s)e^{-\mu s} \frac{Q_0(T)\alpha(T)}{Q_0(s)} e^{-\mu(T-s)}(V-v),
$$

or may prefer to buy at time T for a profit of $f(T)\alpha(T)e^{-\mu T}(V-v)$. Then if all buyers buy upon arrival $Q_0(s) = f(s)$ and $\alpha(s) = 1$, then the previous quantity is actually the constant $Q_0(T)\alpha(T)e^{-\mu T}(V-v)$, which in particular is maximized at time $s = t$. Therefore, $f(s) = Q_0(s)$ is an equilibrium for the proposed pricing p^*. By denoting $G^* = 1 - Q_0$, we conclude that $(p^*, G^*) \in E$.

Observe that by subtracting an arbitrarily small quantity $\varepsilon > 0$ to the price function above we can make that the utility of each buyer is strictly maximized at time $s = t$. Indeed this extra term ε will be multiplied by $e^{-\mu t}Q_0(t)$, so the expected utility of a buyer deciding to buy at time s will be the constant above plus $\varepsilon e^{-\mu s}Q_0(s)$, which is strictly decreasing. This implies that buying upon arrival (i.e., $f(t) = Q_0(t)$), is an equilibrium in which all buyers strictly prefer their choice. Of course, as ε is arbitrarily small, this price function achieves a profit for the seller which is arbitrarily close to that obtained with p^*.

Now, we compute the expected revenue for the seller of the pricing policy p^*. This revenue calculation follows exactly as in (6) and doing integration by parts after this. It follows that the revenue of the seller is given by:

$$
\tilde{\pi}(p^*,G^*) = V(1-Q_0(T)) + Q_0(T)\alpha(T)(V-v)\left(\ln(Q_0(T)) - e^{-\mu T}\mu \int_0^T e^{\mu t} \ln(1-G^*(t))dt \right). \quad (8)
$$

Finally, to connect the quantities in (7) and (8) we need to establish a relation between the last integrals in both terms. Note that since for every $(p, G) \in E$ we have that $1 - G \geq Q_0$, we can bound any such distribution as $G \leq G^*$. Therefore, using the monotonicity of the function $\ln(\cdot)$, we deduce from (7) that for all equilibrium $(p, G) \in E$:

$$\begin{aligned}\tilde{\pi}(p, G) &\leq V(1 - Q_0(T)) + Q_0(T)\alpha(T)(V - v)\left(\ln(Q_0(T)) - e^{-\mu T}\mu \int_0^T e^{\mu t}\ln(1 - G(t))dt\right). \\ &\leq V(1 - Q_0(T)) + Q_0(T)\alpha(T)(V - v)\left(\ln(Q_0(T)) - e^{-\mu T}\mu \int_0^T e^{\mu t}\ln(1 - G^*(t))dt\right). \\ &= \tilde{\pi}(p^*, G^*).\end{aligned}$$

Hence, (p^*, G^*) is the optimal pricing function and associated equilibrium.

3.3. **Best Pricing Strategy.** We are now ready to compare the revenues of the candidates to best pricing policy. Recall that this is either the *Constant pricing* or the best possible *Markdown*. To make the calculation more specific note that we are assuming non-homogeneous Poisson arrivals so that, if we let $m(t) = \int_0^t \Lambda(s)ds$ and $\ell(t) = \int_0^t \lambda(s)ds$, we have $Q_0(T) = e^{-m(T)}$, $Q_1(T) = m(T)e^{-m(T)}$, $q_0(T) = e^{-\ell(T)}$, $q_1(T) = \ell(T)e^{-\ell(T)}$, and $\alpha(T) = \sum_{k=0}^{\infty} q_k(T)/(1+k) = (1 - e^{-\ell(T)})/\ell(T)$.

Constant Pricing: Observe that in the constant price strategy the seller obtains value V if and only if at least one high value buyer arrives. Thus her expected revenue is:

$$V(1 - Q_0(T)) = V(1 - e^{-m(T)}). \tag{9}$$

Markdown: To evaluate the revenue of the optimal Markdown pricing strategy we use equations (4) and (8). These lead to an expected revenue of:

$$\begin{aligned}& V(1 - Q_0(T)) + Q_0(T)\alpha(T)(V - v)\left(\ln(Q_0(T)) - e^{-\mu T}\mu \int_0^T e^{\mu t}\ln(1 - G^*(t))dt\right) \\ & +vQ_0(T)(1 - q_0(T)) \\ = \quad & V(1 - Q_0(T)) + e^{-m(T)}(1 - e^{-\ell(T)})\left(v - \frac{V - v}{\ell(T)}\left(m(T) - e^{-\mu T}\mu \int_0^T e^{\mu t}m(t)dt\right)\right). \end{aligned} \tag{10}$$

In summary the markdown strategy is better if and only if

$$v\left(\ell(T) + m(T) - \mu \int_0^T e^{-\mu(T-t)}m(t)dt\right) > V\left(m(T) - \mu \int_0^T e^{-\mu(T-t)}m(t)dt\right),$$

which may be rewritten as

$$\frac{V}{v} < \left(1 + \frac{\ell(T)}{m(T) - \mu \int_0^T e^{-\mu(T-t)}m(t)dt}\right). \tag{11}$$

Observe that, as one may intuitively expect, this condition is invariant under rescaling of the valuations. What is probably less intuitive is that this condition is also invariant under rescaling of the arrival rates. This happens since under such a rescaling the price path p^* gets closer to the value V, which is possible since the threat of low valuation buyers becomes more powerful.

Interestingly in this situation without discount rate, i.e., $\mu = 0$, our price path obtains the same revenue as an optimal mechanism where all bidders arrive at time 0. This constitutes a stronger version of revenue equivalence which applies to random number of bidders (similar to the results of Levin and Ozdenoren (2004) and Haviv and Milchtaich (2012)). Indeed, as pointed out by Skreta (2006), one can argue that the optimal auction in our setting can be derived as follows. Consider that a random number of bidders, distributed as a poisson random variable of parameter $m(T) + \ell(T)$, participate in the auction. The valuation of the bidders are i.i.d. and equal to v with probability $p = \ell(T)/(m(T) + \ell(T))$, and equal to V with probability $1 - p$. In this situation the virtual valuation is not monotone and therefore Myerson's ironing is needed. It turns out that the

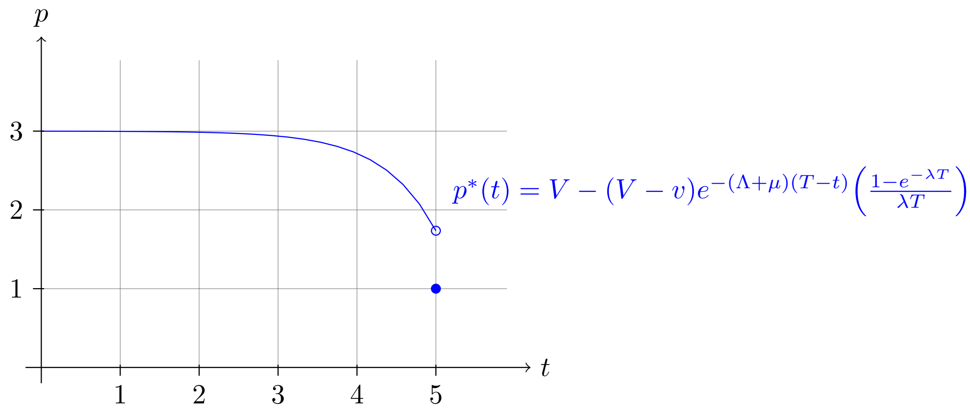


FIGURE 1. Optimal markdown price function for homogeneous Poisson processes of rates $\Lambda = 1$ and $\lambda = 0.2$, and parameters $V = 3$, $v = 1$, $\mu = 0.5$, and $T = 5$.

optimal mechanism can be implemented by:

Case $v > V(1-p)$. Use a fixed price equal to V. Note that the condition of this case is the opposite of (11) (for $\mu = 0$) and, furthermore, the revenue is exactly that given by (9).

Case $v < V(1-p)$. Here the situation is more involved. The optimal mechanism is implemented by running a second price auction (SPA) with reservation price equal to $b = V - (V-v)\frac{1-e^{-\ell(T)}}{\ell(T)}$, but if all bids are below b the item is randomly allocated to any of the participating bidders. The auction is incentive compatible and thus the seller's revenue in this case equals to V times the probability of having two or more high valuation bidders, plus b times the probability of having exactly one high valuation bidder, plus v times the probability of having no high valuation bidder and at least one with low valuation. That is

$$\begin{aligned} & V(1-e^{-m(T)} - m(T)e^{-m(T)}) + bm(T)e^{-m(T)} + ve^{-m(T)}(1-e^{-\ell(T)}) \\ = \quad & V(1-e^{-m(T)}) + e^{-m(T)}(1-e^{-\ell(T)})\left(v - (V-v)\frac{m(T)}{\ell(T)}\right), \end{aligned}$$

exactly as is (10).

Therefore the nonnegative term $\frac{1-e^{-\ell(T)}}{\ell(T)}e^{-m(T)}(V-v)e^{-\mu T}\mu\int_0^T e^{\mu t}m(t)dt$ may be seen as the additional revenue obtained by the seller by exploiting the impatience of the consumers. It is worth mentioning that our pricing scheme can get up to twice the revenue of the optimal static mechanism, and not more than that. To see this note that the best case for our pricing occurs when μ is very large. In this case p^* is essentially constant equal to V and drops to v in the very last minute, implying a revenue of $V(1-e^{-m(T)}) + ve^{-m(T)}(1-e^{-\ell(T)})$. On the other hand, the optimal mechanism gets the revenue expressed in the previous cases. A tedious, but straightforward, calculation shows that this ratio is at most 2, and this can be attained (in the limit) using for instance $V = 1, v = \sqrt{\varepsilon}, m(T) = \varepsilon, \ell(T) = \sqrt{\varepsilon}$. Indeed, in this case the revenue of the optimal mechanism is $1 - e^{-\varepsilon} \approx \varepsilon$ while our pricing mechanisms obtains $1 - e^{-\varepsilon} + e^{-\varepsilon}(1-e^{-\sqrt{\varepsilon}})\sqrt{\varepsilon} \approx 2\varepsilon$.

To finish this section we plot in Figure 1 the optimal pricing path p^* when the rate of the arrival process is constant. Interestingly, the price remains relatively constant until close to the end of the season, where it drops steeply. This seems consistent with the common practice in retail and other industries where aggressive markdown strategies are used. In our setting the optimal price is even discontinuous at time T. This is due to the discrete valuations we consider, however, the phenomenon of steep discounts is prevalent to more general distributions of valuations as we show in the next section.

4. Continuous valuation

In this section we relax the assumption of having just two possible valuations for the item and consider the general case in which the buyers' valuation for the item are i.i.d. distributed according to a continuous distribution $\Phi\colon [v, V] \to [0, 1]$ with associated density ϕ. Therefore, we assume that buyers arrive according to a non-homogeneous Poisson process of rate $\Lambda(\cdot)$ and the probability that a buyer arriving at time t has a valuation less than or equal to v is $\Phi(v)$.

The main result of this section is that, under an assumption over the equilibrium strategies, we reduce the seller's problem to solving a system of ordinary differential equations (ODE). To this end we first show that we can reduce to equilibria taking the form of a threshold $\varphi(\cdot)$, implying that a buyer arriving a time t with valuation u will buy upon arrival if $u > \varphi(t)$, will buy at time $s \in \varphi^{-1}(u)$ (with $s \geq t$) if $u \leq \varphi(t)$, and will not buy if such a time does not exist. Second we prove that a first order approach is sufficient to write the seller's optimization problem. Finally, using optimal control theory, we write down the seller's problem in a way can can be dealt with numerically.

4.1. **Threshold strategies.** Following the notation is Section 2, given a price function p, we let f be a corresponding equilibrium with associated distributions $H = (H_t^u)_{t\in[0,T]}^{u\in[v,V]}$. Here, H_t^u is a probability distribution over $[t, T]$ corresponding to the (mixed) strategy of a buyer arriving at time t with valuation u. Observe that, for a continuous price function, the random allocation probability $\alpha(t)$ will always be 1 and the probability that the item is available at time t, $f(t)$, is a continuous function. Thus, we have that $\mathrm{supp}(H_t^u) \subseteq \arg\max_{t\leq s\leq T}(u - p(s))e^{-\mu s} f(s)$. A key *monotonicity property* is that if we consider an equilibrium, and two valuations $u < u'$, then for all $s \in \mathrm{supp}(H_t^u)$ with $s > t$ and $s' \in \mathrm{supp}(H_t^{u'})$, we have that $s > s'$. Indeed, assume s is the smallest element in $\mathrm{supp}(H_t^u)$ and write the utility of a buyer with utility u' buying at time τ as

$$(u' - p(\tau))e^{-\mu\tau} f(\tau) = (u - p(\tau))e^{-\mu\tau} f(\tau) + (u' - u)e^{-\mu\tau} f(\tau). \tag{12}$$

Since the second term is decreasing and the first is maximized at $\tau = s$, the whole utility is maximized at a point $s' \in [0, s)$.[2] This monotonicity property implies that if a buyer arriving at t buys upon arrival, then any higher valuation buyer arriving at t will also buy upon arrival.

With the monotonicity property at hand it is immediate that for an equilibrium (p, f) with associated distributions H, the function $\varphi(t) = \inf\{u \mid t \in \mathrm{supp}(H_t^u)\}$[3] defines a threshold with the desired property. To see this note that equivalently φ may be defined as

$$\varphi(t) = \inf\{u \mid t \in \arg\max_{t\leq s\leq T}(u - p(s))e^{-\mu s} f(s)\},$$

so that a buyer arriving at t has valuation $u > \varphi(t)$ will buy immediately and otherwise will wait until a time s for which $u = \varphi(s)$.

As the reader could realize, the monotonicity property imposes a certain order in the equilibrium strategies which is summarized by the threshold characterization. From a mechanism design perspective, the threshold function is inherently connected to the allocation rule associated to the mechanism of a posted price p. In fact, if the threshold function turns out to be non-increasing, the allocation rule consists of giving the item to the player with minimum $\tau = \max\{t, \varphi^{-1}(u)\}$, where (t, u) is the respective type of the player.

Moreover, a violation of this non-increasing threshold property, would imply scenarios (with positive probability of occurrence) where a player u arriving at t received the item but if she arrived shortly afterwards she waits to purchase at time $t + c$. Thus, the chances of obtaining the item depends that no one with higher valuation arrives between $[t, t + c]$. This strange situation let us to conjecture that in the optimal pricing the threshold must be non-increasing. Unfortunately

[2]Clearly $s' \in [0, s]$ and some basic calculus shows that actually $s' < s$.

[3]We define the infimum over the empty set as V.

we have been unable to formally prove the latter. Nevertheless, under some conditions, e.g. if the sets $S_u := \arg\max_{s\geq 0}(u - p(s))e^{-\mu s}f(s)$ are connected, one can indeed prove that the threshold equilibrium is non-increasing. For the rest of the section we make the following assumption.

Assumption 1. *In the revenue maximizing pricing policy, the buyers' equilibrium is characterized by a non-increasing threshold function.*

Thus, for sorting out the seller's problem we restrict the attention to this class of equilibria. In what follows we exploit this conjecture to simplify the seller's optimization problem. An important implication of our assumption is that if φ is non-increasing there are at most countably many valuations u for which the pre-image $\varphi^{-1}(u)$ is not a singleton. Therefore, for almost all valuations u and arrival times t a buyer with valuation u and arriving at time t will buy at time $\max\{t, \varphi^{-1}(u)\}$. In conclusion, almost all players are playing pure strategies.

4.2. **First order approach.** We now consider a price function with a non-increasing threshold equilibrium, which we denote by (p, φ) and assume that both p and φ are differentiable. The goal of this section is to show that the first order approach is sufficient to deal with the seller's problem.

Let us first write down the seller's problem. Recalling that $m(t) = \int_0^t \Lambda(s)ds$ is the average arrival rate until time t, denoting by E the set of equilibrium pairs[4] (p, φ) such that φ is non-increasing, and noting that $f(t) = e^{-m(t)(1-\Phi(\varphi(t)))}$, since it equals the probability of having no arrivals in $[0, t]$ with valuation above $\varphi(t)$, the seller's problem can be written as

$$\max_{p:(p,\varphi)\in E} \int_0^T -\frac{d}{ds}\big(e^{-m(s)(1-\Phi(\varphi(s)))}\big)p(s)ds. \tag{13}$$

Denoting by $U(u,t) := (u-p(t))e^{-\mu t}f(t) = (u-p(t))e^{-\mu t}e^{-m(t)(1-\Phi(\varphi(t)))}$ the utility (at equilibrium) of a buyer with valuation u buying at time t, checking that $(p, \varphi) \in E$ amounts to checking that $U(u,t)$ is maximized at $t = \varphi^{-1}(u)$. On the other hand the first order optimality conditions restrict to checking that $\partial_2 U(u, \varphi^{-1}(u)) = 0$. Similarly to the monotonicity property we compute

$$\partial_2 U(u,t) = -p'(t)e^{-\mu t}f(t) + p(t)\mu e^{-\mu t}f(t) - p(t)e^{-\mu t}f'(t) - ue^{-\mu t}(\mu f(t) - f'(t)), \tag{14}$$

and note that evaluating at $t = \varphi^{-1}(u)$ the previous quantity is zero. Then, since the term $\mu f(t) - f'(t) > 0$, $\partial_2 U(u', \varphi^{-1}(u)) < 0$ whenever $u' > u$ and $\partial_2 U(u', \varphi^{-1}(u)) > 0$ whenever $u' < u$. Equivalently, since φ is non-increasing, we have that

$$\partial_2 U(u,t) \begin{cases} > 0 & \text{if } t < \varphi^{-1}(u) \\ = 0 & \text{if } t = \varphi^{-1}(u) \\ < 0 & \text{if } t > \varphi^{-1}(u). \end{cases}$$

Thus, just with the first order optimality conditions we know that as a function of t, $U(u,t)$ increases until $t = \varphi^{-1}(u)$ and then it decreases, implying that $\varphi^{-1}(u)$ is a global maximizer. We have thus established that (13) is equivalent to

$$\max_{p,\varphi:\,\partial_2 U(u,\varphi^{-1}(u))=0,\,\varphi'\leq 0} \int_0^T -\frac{d}{ds}\big(e^{-m(s)(1-\Phi(\varphi(s)))}\big)p(s)ds. \tag{15}$$

4.3. **Solving the sellers problem using optimal control.** Under Assumption 1, we now apply optimal control theory to solve problem (15). It is worth mentioning that although we must restrict our attention to a differentiable setting, smooth functions are dense in the continuous functions space, so our method obtains numerical solution to the seller's problem in this more general setting.

[4]Note that equilibrium is actually defined as a price and a probability of availability f, but using (12), we see that φ characterizes the equilibrium as well.

To transform (15) to the classic optimal control setting we observe that at an equilibrium (p, φ) we must have that $p(T) = \varphi(T)$. Then, writing f explicitly in (14), we have that (15) becomes

$$\max_{\alpha,p,\varphi} \int_0^T -\frac{d}{ds}\big(e^{-m(s)(1-\Phi(\varphi(s)))}\big)p(s)ds$$
$$\text{s.t.} \begin{cases} -p' = (\varphi - p)(\mu + m'(s)(1-\Phi(\varphi)) - m(s)\phi(\varphi(s))\varphi'(s)) & \text{for } s \in (0,T) \\ p(T) = \varphi(T) = \alpha \\ \varphi' \le 0. \end{cases}$$

In order to solve this problem, we introduce the auxiliary functions $q(s) = p(T-s)$ and $\psi(s) = \varphi(T-s)$, for every $s \in [0,T]$. By using the change of variables $\tau = T - s$, the problem becomes

$$\max_{\alpha,q,\psi} I := \int_0^T \frac{d}{d\tau}\big(e^{-m(T-\tau)(1-\Phi(\psi(\tau)))}\big)q(\tau)d\tau$$
$$\text{s.t.} \begin{cases} q' = (\psi - q)(\mu + m'(T-\tau)(1-\Phi(\psi)) + m(T-\tau)\phi(\psi)\psi') & \text{for } \tau \in (0,T) \\ q(0) = \psi(0) = \alpha \\ \psi' \ge 0. \end{cases} \tag{16}$$

Note that, from the differential equation on q we obtain

$$q(\tau) = \psi(\tau) - e^{\mu(T-\tau)+m(T-\tau)(1-\Phi(\psi(\tau)))} \int_0^\tau \psi'(s)e^{-\mu(T-s)-m(T-s)(1-\Phi(\psi(s)))}ds.$$

Using integration by parts, this yields

$$\begin{aligned} I = &\int_0^T \frac{d}{d\tau}\big(e^{-m(T-\tau)(1-\Phi(\psi(\tau)))}\big)\psi(\tau)d\tau \\ &\qquad + \int_0^T \frac{d}{d\tau}\big(m(T-\tau)(1-\Phi(\psi(\tau)))\big)e^{\mu(T-\tau)} \int_0^\tau \psi'(s)e^{-\mu(T-s)-m(T-s)(1-\Phi(\psi(s)))}dsd\tau \\ &= \psi(T) - \alpha e^{-m(T)(1-\Phi(\alpha))} - \int_0^T e^{-m(T-\tau)(1-\Phi(\psi(\tau)))}\psi'(\tau)d\tau \\ &\qquad - \int_0^T m(T-\tau)(1-\Phi(\psi(\tau)))e^{\mu(T-\tau)}\Big(\psi'(\tau)e^{-\mu(T-\tau)-m(T-\tau)(1-\Phi(\psi(\tau)))} - \mu r(\tau)\Big)d\tau, \end{aligned}$$

where r is the solution to the ordinary differential equation

$$\begin{cases} r' = \psi' e^{-\mu(T-\tau)-m(T-\tau)(1-\Phi(\psi))} & \text{for } \tau \in (0,T) \\ r(0) = 0. \end{cases}$$

Therefore, by setting $u = \psi'$, (16) is equivalent to

$$\min_{\alpha,q,\psi} \int_0^T \ell(\tau, \psi(\tau), r(\tau), u(\tau))d\tau + R(\psi(T), r(T))$$
$$\text{s.t.} \begin{cases} r' = ue^{-\mu(T-\tau)-m(T-\tau)(1-\Phi(\psi))} & \text{for } \tau \in (0,T) \\ r(0) = 0 \\ \psi' = u \ge 0 \quad \text{for } \tau \in (0,T) \\ \psi(0) = \alpha, \end{cases} \tag{17}$$

where

$$\begin{cases} \ell(\tau, \psi, r, u) := e^{-m(T-\tau)(1-\Phi(\psi))}u(1 + m(T-\tau)(1-\Phi(\psi))) - \mu r m(T-\tau)(1-\Phi(\psi))e^{\mu(T-\tau)} \\ R(\psi, r) := \alpha e^{-m(T)(1-\Phi(\alpha))} - \psi. \end{cases}$$

Note that (17) is a classical optimal control problem, where u is the control and (ψ, r) is the state. Hence, we deduce that any solution of this problem must satisfy the first order necessary conditions (Pontryagin's minimum principle, see e.g. (Vinter, 2000, Section 6.2))

$$(\forall \tau \in]0,T[) \quad \begin{cases} u(\tau) \in \arg\min_{u \in \mathbb{R}_+} H(\tau, \psi(\tau), r(\tau), u, w(\tau), \eta(\tau)) \\ -w'(\tau) = \frac{\partial H}{\partial r}(\tau, \psi(\tau), r(\tau), u(\tau), w(\tau), \eta(\tau)) \\ w(T) = \frac{\partial R}{\partial r}(\psi(T), r(T)) = 0 \\ -\eta'(\tau) = \frac{\partial H}{\partial \psi}(\tau, \psi(\tau), r(\tau), u(\tau), w(\tau), \eta(\tau)) \\ \eta(T) = \frac{\partial R}{\partial \psi}(\psi(T), r(T)) = -1, \end{cases} \tag{18}$$

where H is the Hamiltionian of the system

$$H(\tau, \psi, r, u, w, \eta) = \ell(\tau, \psi, r, u) + wue^{-\mu(T-\tau)-m(T-\tau)(1-\Phi(\psi))} + \eta u. \tag{19}$$

Thus we have transformed the seller's problem to solving (17)-(18), a system of four ordinary differential equations with initial value, coupled with a hamiltonian equation.

4.4. **Numerical experiments.** For solving system (17)-(18) numerically, we discretize the interval $[0,T]$ into $N_h = T/h$ subintervals of length h, starting from a given piecewise linear control function u_h^0. For every $n \in \mathbb{N}$, we find piecewise linear functions r_h^n and ψ_h^n by solving the differential equations in (17) via a forward Euler's method and we find w_h^n and η_h^n by solving the differential equations in (18) via a backward Euler's method. Then we obtain u_h^{n+1} by computing the projected gradient step

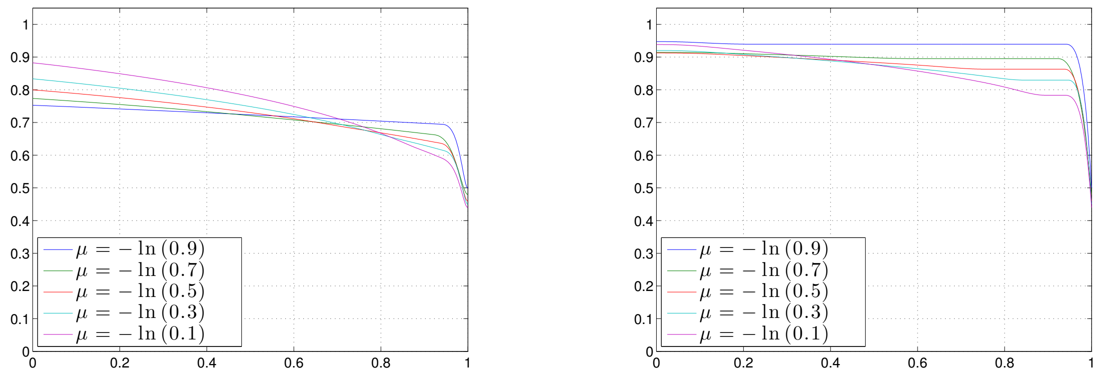
$$u_h^{n+1}(\tau_i) = P_{\mathbb{R}_+}\Big(u_h^n(\tau_i) - \gamma \frac{\partial H}{\partial u}(\tau_i, \psi_h^n(\tau_i), r_h^n(\tau_i), u_h^n(\tau_i), w_h^n(\tau_i), \eta_h^n(\tau_i))\Big), \tag{20}$$

where, for $i = 0, \dots, N_h$, we let $\tau_i = i \cdot h$, the parameter $\gamma > 0$ is chosen appropriately, and $P_{\mathbb{R}_+}(x) = \max\{0, x\}$. The algorithm stops when $\max_{0 \le i \le N_h} |u_h^{n+1}(\tau_i) - u_h^n(\tau_i)| < \varepsilon$, for $\varepsilon > 0$ small enough. All our computations consider that Φ is the uniform distribution in $[0,1]$ and $T = 1$. The parameters are set to be $\varepsilon = 0.005$, $h = 0.001$, and $\gamma = 0.3$.

In Figure 2 we vary the discount rate while the arrivals are modeled via an homogeneous Poisson process of fixed rate $\lambda = 3$. On the other hand, in Figure 3 we fix $\mu = -\ln(0.7)$ and vary the arrival rate of the buyers. Table 1 and Table 2 exhibit the profits obtained in each case and compare it to that of an optimal auction (that takes place at the end of the season). We verify that the profit obtained with the markdown strategy is always better than that of the optimal auction and that, quite naturally, the difference increases when μ increases. Maybe not so naturally, note that when μ is large, it is more profitable to decrease the reserve price. The situation with fixed discount rate and varying λ is different. There, it is not clear that a higher arrival rate impacts the profit ratio. It is natural however that the reservation price is not significantly affected by the number of buyers as this is also the case in an optimal auction.

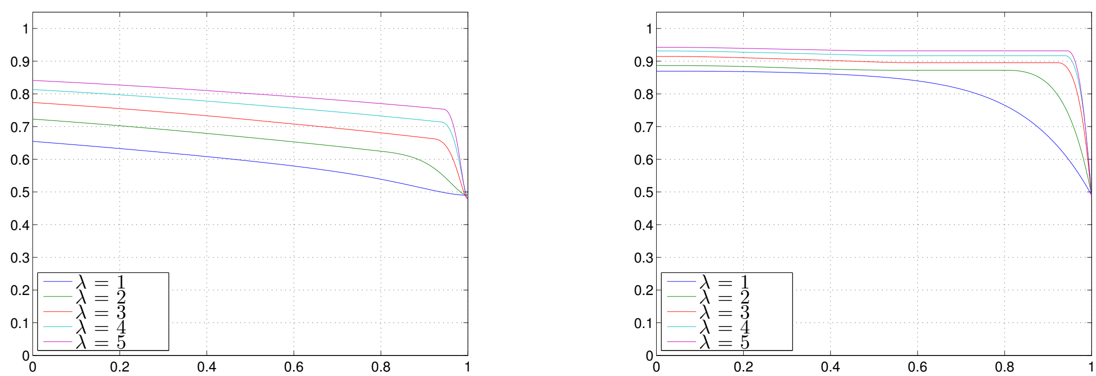
Secondly, the numerical results shows that in the continuous model the reservation price $p(T)$ is affected by the temporal discount rate; in contrast to the discrete model where we proved that the reservation price is invariant of the discount factor. The intuition behind this fact is that in the discrete model, since there are only two valuation types, the reservation price is just used to avoid that higher-valuation buyers decide to go the lottery at the end the season. In contrast, in the continuous setting, the reservation price is used to split the bidders that are ex-ante interesting (for the seller) to trade with. Since the valuation is affected by time with the discount rate, it is quite natural that here the reservation price depends on the discount factor.

Another interesting observation is that in our numerical simulations, as in Section 3, when $\mu = 0$, we recover the classic result of static optimal mechanism design when we do not consider a temporal discount factor. Indeed as μ approaches 0 the optimal price function becomes flat and decreases very quickly to 0.5 at time 1. This is a way of simulating a first price auction at time 1 with



(A) Optimal pricing policy. (B) Threshold equilibrium.

FIGURE 2. Optimal price and threshold for homogeneous Poisson arrivals of rate $\lambda = 3$.



(A) Optimal pricing policy. (B) Threshold equilibrium.

FIGURE 3. Optimal price and threshold functions for a discount factor $\mu = -\ln(0.7)$.

a reserve price of 0.5, which is known to yield the optimal revenue. For instance, taking $\lambda = 3$ and $\mu = -\ln(0.99)$ we obtain that the optimal price is essentially 0.740 until time 0.95 (with an underlying threshold of value essentially 0.99). This is in almost perfect match with the fact that in a first price auction with a random number of bidders distributed poisson of rate 3 the maximum possible bid, i.e., that of a bidder with valuation 1 is equal to 0.741.

5. Concluding remarks

We have studied a two stage dynamic game where in the first stage a seller proposes a markdown path price for selling a single item, while strategic consumers respond to this by selecting the optimal time to buy the item considering the risk of not getting it.

In particular, we have characterized the optimal price function when buyers valuations can only take two values. Interestingly, this function satisfies an important economic property: it is incentive compatible. Indeed, even if the seller cannot observe the buyers arrival time (a common situation in practice) it is in the buyers' best interest to buy upon arrival, thus revealing their private type. Furthermore, the revenue obtained by this price function is at least as large as that of the optimal mechanism in this context. In this respect, the obtained optimal price function is discontinuous at

μ	Profit	α	Opt. Auction	Ratio
$-\ln(0.9) = 0.105$	0.4877	0.50	0.4821	1.011
$-\ln(0.7) = 0.357$	0.4900	0.48	0.4821	1.016
$-\ln(0.5) = 0.693$	0.4973	0.46	0.4821	1.031
$-\ln(0.3) = 1.204$	0.5072	0.45	0.4821	1.052
$-\ln(0.1) = 2.303$	0.5230	0.44	0.4821	1.085

TABLE 1. Optimal profits when $\lambda = 3$ comparison with the revenue of an optimal auction at the end of the season.

λ	Profit	α	Opt. Auction	Ratio
1	0.2166	0.49	0.2131	1.016
2	0.3733	0.49	0.3679	1.015
3	0.4900	0.48	0.4821	1.016
4	0.5777	0.49	0.5677	1.018
5	0.6450	0.48	0.6328	1.019

TABLE 2. Optimal profits when $\mu = -\ln(0.7)$ and comparison with the revenue of an optimal auction at the end of the season.

the end of the season which nicely mimics the implicit random allocation necessary in the optimal mechanism.

We also derive an numerical approach to tackle the general valuation case. Our numerical results show in particular that the fact that buyers discount the future faster than the seller severely affect the optimal pricing policy. Indeed one can easily derive from the work of Gershkov et al. (2014) that if both the seller and the buyers discount the future equally then the optimal mechanism takes a form of a threshold (as in our case) which turns out to be constant until the end of the season, time at which an auction is run. In our case however the threshold is far from constant for small arrival rates or large discount rates.

Throughout the paper we have assumed that the seller has commitment power and can credibly preannounce a certain price path. However, as we deal with the single unit case this assumption is not really needed. Indeed when there is a single unit on sale the optimal preannounced price function and the optimal *dynamic* price function (in which the seller does not make commitments) actually coincide. Therefore all our results apply to the case without commitment as well.

Finally, important extensions that require further investigation are to characterize the optimal path price function in more general frameworks including, when consumers have a random private value over a continuous distribution, and when there are multiple units to be sold.

Acknowledgments. We would like to thank Gustavo Vulcano for his valuable comments on a earlier draft of this paper. We also thank the three reviewers and the associate editor for several suggestions that greatly improved the paper.

References

Aviv, Y., A. Pazgal. 2008. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing & Service Operations Management* 10(3):339–359.

Board, S., A. Skrzypacz. 2015. Revenue management with forward-looking buyers. *Journal of Political Economy*, to appear.

Cachon, G., P. Feldman. 2010. Dynamic versus static pricing in the precense of strategic consumers. *Manuscript.*

Caldentey, R., G. Vulcano. 2007. Online auction and list price revenue management. *Management Sience* 53(5):795–813.

Caldentey, R., Y. Liu, I. Lobel. 2013. Intertemporal pricing without priors. *Manuscript*.

Correa, J.R., R. Montoya, C. Thraves. 2012. Contingent preannounced pricing policies with strategic consumers. *Manuscript*.

Elmaghraby, W., S. Lippman, C. Tang, R. Yin. 2009. Pre-announced pricing strategies with reservations. *Production and Operations Management* 18:381–401.

Gershkov, A., B. Moldovanu, P. Strack. 2014. Dynamic allocation and learning with strategic arrivals. *Manuscript*.

Haviv, M., I. Milchtaich. 2012. Auctions with a random number of identical bidders. *Economic Letters* 114(2):143–146.

Jerath, K., S. Netessine, S.K. Veeraraghavan. 2010. Revenue management with strategic customers: Last-minute selling and opaque selling. *Management Science* 56(3):430–448.

Levin, D., E. Ozdenoren. 2004. Auctions with uncertain numbers of bidders. *Journal of Economic Theory* 118:229–251.

Levin, D., J.L. Smith. 1994. Equilibrium in auctions with entry. *The American Economic Review* 84:585–599.

Li, J., N. Granados, S. Netessine. 2014. Are consumers strategic? structural estimation from the air travel industry. *Management Science* 60(9): 2114–2137.

Liu, Q., G. van Ryzing. 2011. Strategic capacity rationing when customers learn. *Manufacturing & Service Operations Management* 13(1):89–107.

McAfee, R.P., J. McMillan. 1987. Auctions with a stochastic number of bidders. *Journal of Economic Theory* 43:1–19.

Mierendorff, K. 2011. Optimal dynamic mechanism design with deadlines. *Manuscript*.

Surasvadi, N., G. Vulcano. 2013. Price matching under conditional discounts in the presence of strategic consumers. *Manuscript*.

Osadchiy, N., G. Vulcano. 2010. Selling with binding reservations in the presence of strategic consumers. *Management Science* 56(12):2173–2190.

Pai, M., R. Vohra. 2013. Optimal dynamic auctions and simple index rules. *Mathematics of Operations Research* 38(4):682–697.

Russo, R., G. Shmueli, W. Jank, N. Shyamalkumar. 2010. Models for Bid Arrivals and Bidder Arrivals in Online Auctions. Encyclopedia of Statistical Sciences, Chapter 23, Wiley, New York.

Skreta, V. 2006. Mechanism design for arbitrary type spaces. *Economic Letters* 91(2):293–299.

Su, X. 2007. Inter-temporal pricing with strategic customer behavior. *Management Science* 53(5):726–741.

Talluri, K., G. van Ryzin. 2005. The Theory and Practice of Revenue Management. Springer, New York.

Vinter, R., 2000. Optimal Control. Birkhäuser, Berlin.

Yin, R., Y. Aviv, A. Pazgal, C. Tang. 2009. Optimal markdown pricing: Implications of inventory display formats in the presence of strategic customers. *Management Science* 55(8):1391–1408.

Appendix A. Existence of equilibrium

In this section we prove that for a large family of *reasonable* price functions of the seller there is an equilibrium among the buyers with high valuation. As low valuation buyers do not behave strategically (because the seller does not have incentives to lower the price below v) this shows existence of equilibrium for our problem. Specifically we prove:

Theorem 2. *Consider two classes of buyers arriving according to non-homogeneous poisson processes with continuous arrival rate; the first class has rate $\Lambda\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ and value the item at V, the second group has rate $\lambda\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ and value the item at v. If the seller commits to a price function $p\colon [0,T) \to (v,V)$ which is continuous on $[0,T)$, satisfying $\lim_{t\to T^-} p(t) \in [v, b^*]$, where $b^* = V - \frac{(V-v)(1-e^{-\int_0^T \lambda(t)dt})}{\int_0^T \lambda(t)dt}$, and $p(T) = v$, then there exists a symmetric equilibrium.*

Observe that we are considering the case of a Markdown strategy assuming in addition that $p(t) < V$. This is without loss of generality since if $p(t) = V$ for some $t \in [0,T]$, $U(t) = 0$ for the buyers and thus nobody buys at this time (unless $p \equiv V$, the case of constant pricing).

The technical requirement on $p(t)$ close to T is just to ensure that a high-value buyer does not prefer to wait until the deadline and participate in the lottery with low-value costumers.

It is worth mentioning that the standard fixed point approaches to prove existence of equilibrium do not seem work here since we have infinitely many players with an infinite set of available pure strategies. Furthermore, the natural fixed-point mapping is hard to analyze. Thus to prove Theorem 2 we take a constructive approach and build an equilibrium which is in a way symmetric.

The technique for characterizing the equilibrium is innovative. Solving the trade-off between waiting for a lower price and risking of loosing the item, induces to split the season into two disjoint subsets. One where the buyers' strategy consist to buy upon arrival and the other set where buyers use mixed strategies. Consequently, the first main idea of our construction is to divide the interval $[0,T]$ into subintervals. In some of these subintervals buyers will simply buy upon arrival while in others they will use a mixed strategy over that subinterval. In the subintervals where mixed strategies are used, the conditional distribution determining the buying time of a consumer that arrived at time t and has already waited until time s, is independent of t. Thus, all consumers that wait until a certain time, behave identically. Is in this sense that our constructed equilibrium is symmetric.

The second main idea of the proof is the construction of this symmetric equilibrium within an interval, (t_1, t_2), in which mixed strategies are used. Here, we first *iron* the price function and show that one can assume that the function $t \mapsto (V - p(t))e^{-\mu t}$ is non-decreasing. Then we impose that a symmetric equilibrium takes the form of a distribution $H\colon [t_1,t_2] \to [0,1]$ which defines the equilibrium strategy of a consumer arriving at time t as the conditional distribution $(H_t)_{t\in[t,t_2]}$. Finally, by imposing the equilibrium conditions on this family, particularly that the whole interval maximizes the utility of a buyer, we are able to characterize this distribution through a differential equation, whose solution somewhat surprisingly satisfies all desired properties.

By the technical requirement mentioned before, the main difficulty of the proof consists in developping strategies that avoid deviations over $[0,T)$. For this reason, we first construct equilibrium strategies assuming that p is continuous over $[0,T]$, with $p(T) \in [v, b^*]$ and at the end of the section we show that the same strategies sustain an equilibrium for the case stated in Theorem 2.

A.1. **Time horizon decomposition.** We now decompose the interval $[0,T]$ into subintervals. The key properties of these sub intervals is that all consumers will actually buy in the subinterval they arrived.

Given a continuous function $p\colon [0,T) \to (v,V)$ such that $p(T) \in [v,b^*]$, and letting for all $t \in [0,T]$ the average arrival rate until time t, $m(t) = \int_0^t \Lambda(x)dx$, we consider the set $I \subseteq [0,T]$:

$$I := \Big\{t \in [0,T] : t \in \arg\max_{s\in[t,T]} (V - p(s))e^{-(\mu s+m(s))}\Big\}. \tag{21}$$

In our constructed equilibria, buyers arriving in I will buy upon arrival. Consider now

$$t_* \;:=\; \min\Big\{t \in [0,T] : t \in \arg\max_{s\in[0,T]} (V - p(s))e^{-(\mu s+m(s))}\Big\} \tag{22}$$

$$t_0 \;:=\; \min\big\{t \in [0,t_*] : (V - p(t))e^{-\mu t} = (V - p(t_*))e^{-(\mu t_*+m(t_*))}\big\}. \tag{23}$$

Lemma 3. *The quantities t_* and t_0 are well defined. Furthermore $t_* = 0$ if and only if $t_0 = 0$.*

Proof. First note that t_* is well defined since p and m are continuous and $[0,T]$ is compact, implying that the set of maximizers of $t \mapsto (V - p(t))e^{-(\mu t+m(t))}$ is also compact. Observe that if $t_* = 0$ then $t_0 = 0$. Conversely if $t_* > 0$ then $0 \notin \arg\max_{s\in[0,T]}(V - p(s))e^{-(\mu s+m(s))}$, which yields $(V - p(0)) < (V - p(t_*))e^{-(\mu t_*+m(t_*))}$. Moreover, since $m(t_*) > 0$, we have that $(V - p(t_*))e^{-\mu t_*} > (V - p(t_*))e^{-(\mu t_*+m(t_*))}$. Altogether, the continuity of $t \mapsto (V - p(t))e^{-\mu t}$ yields the existence of t_0 and also that $t_* = 0$ if and only if $t_0 = 0$. $\square$

Buyers arriving in $(0, t_*)$ will use a mixed strategy with support in the interval $(t_0, t_*]$. Note that it makes no sense to buy at $t < t_0$, since $U(t) \leq (V - p(t))e^{-\mu t} < (V - p(t_*))e^{-(\mu t_*+m(t_*))} \leq U(t_*)$. The previous inequality is obvious for $t = 0$ by definition of t_* (otherwise $t_* = 0$), and it holds until t_0 by definition of t_0.

As we will prove in Lemma 8, the remainder of the interval $[0,T]$, can be decomposed into a collection of open intervals of the form (t_1, t_2), such that

$$t_2 \;=\; \min\Big\{t \in [0,T] : t \in \arg\max_{s\in(t_1,T]} (V - p(s))e^{-(\mu s+m(s))}\Big\}, \tag{24}$$

and t_1 is the largest $t < t_2$ satisfying

$$(V - p(t_1))e^{-(\mu t_1+m(t_1))} = (V - p(t_2))e^{-(\mu t_2+m(t_2))}. \tag{25}$$

For these intervals, buyers arriving in an interval of the form (t_1, t_2) will buy within the interval according to a mixed strategy defined in the next section. Note also for $t_1 < t < t_2$ we have that $(V - p(t))e^{-(\mu t+m(t))} < (V - p(t_2))e^{-(\mu t_2+m(t_2))}$. We refer to these intervals (t_1, t_2) as well as the interval (t_0, t_*) as *mixing* intervals, since mixed strategies are used.

In the next section we will show that every mixing interval (t_1, t_2) has a corresponding distribution H with support on $[t_1, t_2]$, such that buyers arriving at time $t \in (t_1, t_2)$ will buy at a random time drawn according to H_t, the conditional distribution of H in $[t, t_2]$. We may summarize our constructed equilibrium as follows (see Figure 4):

(i) Consumers arriving in I buy upon arrival.
(ii) Consumers arriving at time $t \in [0, t_0]$ buy at a random time drawn according to the distribution H corresponding to the mixing interval (t_0, t_*).
(iii) Consumers arriving at time $t \in (t_0, t_*)$ buy at a random time drawn according to the conditional distribution H_t corresponding to distribution H of the mixing interval (t_0, t_*).
(iv) Consumers arriving at time $t \in (t_1, t_2)$ of a generic mixing interval buy at a random time drawn according to the conditional distribution H_t corresponding to distribution H of the mixing interval (t_1, t_2).

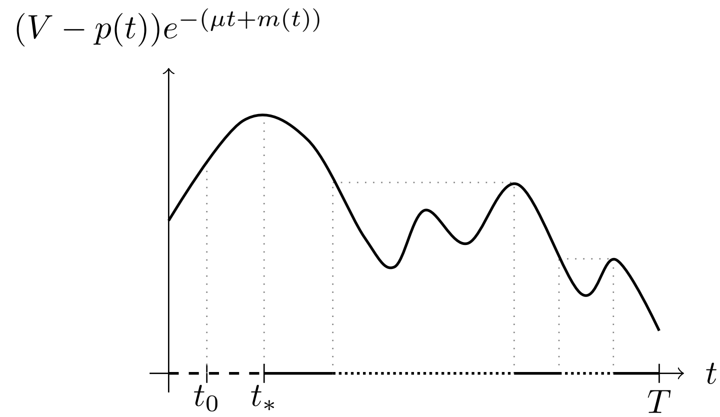


FIGURE 4. Interval $[0, T]$ is decomposed in three collections of intervals. Costumers arriving in I (continuous line) buy straightaway. Costumers arriving in intervals marked by a dotted line play mixed strategies. Finally, costumers arriving in the first interval marked by a dashed line play a mixed strategy with support in (t_0, t_*).

A.2. **Strategy in a mixing interval.** In this section we focus on an mixing interval (t_1, t_2) where t_1 and t_2 satisfy (24) and (25). Indeed we actually isolate (t_1, t_2) assuming that nobody arrived before t_1, which is consistent with our partitioning of the time horizon T. For simplicity we first assume that p is such that the function

$$g_p(t) \quad := \quad (V - p(t))e^{-\mu t}$$

is non-decreasing. At the end of the section we consider an arbitrary continuous price function.

In the following lemma we impose that the mixed strategy of a consumer who arrived at t should be the same as that of those who arrived earlier but did not buy before t (if the item has not been sold). This gives us a closed expression of the availability probability f in terms of H.

Lemma 4. *Let H be a continuous distribution over $[t_1, t_2]$. Assume that buyers arriving at $t \in (t_1, t_2)$ buy according to the conditional distribution of H, then the probability that the item is available at time $t \in [t_1, t_2]$ is given by:*

$$f(t) = \exp\left(- m(t) + (1 - H(t)) \int_{t_1}^{t} \frac{\Lambda(x)dx}{1 - H(x)} \right).$$

Proof. Let h the probability density function of H. Thus the conditional density on $[t, t_2]$ is

$$h_t(s) = \frac{h(s)}{\int_t^{t_2} h(\tau)d\tau}, \text{ for all } t \in (t_1, t_2) \text{ and } s \in [t, t_2]$$

so that the conditional distributions is

$$H_t(s) = \frac{H(s) - H(t)}{1 - H(t)}, \text{ for all } t \in (t_1, t_2) \text{ and } s \in [t, t_2]. \tag{26}$$

To apply equation (1) in the definition of equilibrium we need an expression for the density of the arrival process. For $t \in (t_1, t_2)$ the density of the arrival time in a non-homogeneous Poisson process between (t_1, t) is given by $dF_t(x) = \frac{\Lambda(x)}{m(t)-m(t_1)}dx$. Also, for every $k \in \mathbb{N}$ we have that

$Q_k(t) = e^{-m(t)+m(t_1)}(m(t)-m(t_1))^k/k!$. Hence, it follows that

$$
\begin{aligned}
f(t) &= e^{-m(t_1)}\mathbb{P}(\text{item is available at } t| \text{ is available at } t_1)\\
&= e^{-m(t_1)}\sum_{k\geq 0} Q_k(t)\int_{t_1}^{t}\cdots\int_{t_1}^{t}\prod_{i=1}^{k}\frac{1-H(t)}{1-H(x_i)}dF_t(x_1)\dots dF_t(x_k)\\
&= e^{-m(t_1)}\sum_{k\geq 0} Q_k(t)(1-H(t))^k\Big(\int_{t_1}^{t}\frac{1}{1-H(x)}dF_t(x)\Big)^k\\
&= e^{-m(t_1)}\sum_{k\geq 0} Q_k(t)\Big((1-H(t))\int_{t_1}^{t}\frac{1}{1-H(x)}dF_t(x)\Big)^k\\
&= e^{-m(t_1)}\sum_{k\geq 0} e^{-(m(t)-m(t_1))}\frac{\Big((m(t)-m(t_1))(1-H(t))\int_{t_1}^{t}\frac{1}{1-H(x)}dF_t(x)\Big)^k}{k!}\\
&= e^{-m(t_1)}\exp\Big[(m(t)-m(t_1))\Big(-1+(1-H(t))\int_{t_1}^{t}\frac{1}{1-H(x)}dF_t(x)\Big)\Big]\\
&= \exp\Big(-m(t)+(1-H(t))\int_{t_1}^{t}\frac{\Lambda(x)}{1-H(x)}dx\Big).
\end{aligned}
$$

□

We now turn to give an explicit expression for the strategies of buyers arriving in a mixing interval. For $t \in (t_1, t_2]$ consider the function

$$
H(t) = 1 - \frac{\ln\Big(\frac{g_p(t_1)}{g_p(t)}\Big) + m(t) - m(t_1)}{K\exp\Big(-\int_{t}^{(t_1+t_2)/2}\frac{\Lambda(x)dx}{\ln\Big(\frac{g_p(t_1)}{g_p(x)}\Big)+m(x)-m(t_1)}\Big)}, \tag{27}
$$

where $K > 0$ is a constant to be determined later. The next result shows that this is actually a distribution and that if all consumers buy according to the conditional distribution given by (26), namely

$$
\begin{aligned}
H_t(s) =&\frac{H(s)-H(t)}{1-H(t)}\\
=&1-\exp\Big(-\int_{t}^{s}\frac{\Lambda(x)dx}{\ln\big(\frac{g_p(t_1)}{g_p(x)}\big)+m(x)-m(t_1)}\Big)\frac{\ln\Big(\frac{g_p(t_1)}{g_p(s)}\Big)+m(s)-m(t_1)}{\ln\Big(\frac{g_p(t_1)}{g_p(t)}\Big)+m(t)-m(t_1)},
\end{aligned} \tag{28}
$$

then their utility $U(t)$ is constant in the interval $[t_1, t_2]$.

Lemma 5. *Let p be a price function such that g_p is non-decreasing. Then there exists $K > 0$ such that H, defined by (27), is non-decreasing, continuous, and satisfies that $H(t_2) = 1$. Furthermore, if all consumers buy according to $(H_t)_{t\in(t_1,t_2)}$, the family of distributions defined in (28), their utility satisfies $U(t) = U(t_1)$ for all $t \in [t_1, t_2]$.*

Proof. We proceed backwards by first imposing that the utility is constant throughout the interval and study the implications of this condition. Since we assume that nobody arrived before t_1 we have $f(t_1) = e^{-m(t_1)}$ and the condition $U(t) = U(t_1)$ reads $(V - p(t))e^{-\mu t}f(t) = (V - p(t_1))e^{-\mu t_1}f(t_1)$, which yields

$$e^{m(t_1)}f(t) = \frac{(V-p(t_1))e^{-\mu t_1}}{(V-p(t))e^{-\mu t}} = \frac{g_p(t_1)}{g_p(t)} \quad \text{for all } t \in (t_1, t_2].$$

Note that, since g_p is non-decreasing, f is non-increasing which is consistent with the fact that f represents the probability of the item being available. From Lemma 4 we obtain the equation

$$\frac{g_p(t_1)}{g_p(t)} = \exp\left(-m(t)+m(t_1)+(1-H(t))\int_{t_1}^{t}\frac{\Lambda(x)dx}{1-H(x)}\right),$$

where the unknown is H. This can be rewritten as:

$$\ln\left(\frac{g_p(t_1)}{g_p(t)}\right) + m(t) - m(t_1) = (1-H(t))\int_{t_1}^{t}\frac{\Lambda(x)dx}{1-H(x)}dx \quad \text{for all } t \in (t_1, t_2].$$

Denoting $u\colon t \mapsto \int_{t_1}^{t}\frac{\Lambda(x)dx}{1-H(x)}$, we have $u'(t) = \Lambda(t)/(1-H(t))$ and then we transform the previous integral equation into the differential equation

$$\frac{\Lambda(t)}{\ln\left(\frac{g_p(t_1)}{g_p(t)}\right) + m(t) - m(t_1)} = \frac{u'(t)}{u(t)}.$$

The latter is solved by integrating from $(t_1+t_2)/2$ to t, which leads to

$$\ln(u(t)) - \ln(u((t_1+t_2)/2)) = \int_{(t_1+t_2)/2}^{t}\frac{\Lambda(x)dx}{\ln\left(\frac{g_p(t_1)}{g_p(x)}\right) + m(x) - m(t_1)}.$$

Defining $K = u((t_1+t_2)/2) = \int_{t_1}^{(t_1+t_2)/2}\frac{\Lambda(x)dx}{1-H(x)} > 0$, we obtain that the solution is

$$u(t) = K\exp\left(\int_{(t_1+t_2)/2}^{t}\frac{\Lambda(x)dx}{\ln\left(\frac{g_p(t_1)}{g_p(x)}\right) + m(x) - m(t_1)}\right),$$

and, hence,

$$(\forall t \in (t_1,t_2)) \quad \frac{\Lambda(t)}{1-H(t)} = u'(t) = \frac{K\Lambda(t)\exp\left(\int_{(t_1+t_2)/2}^{t}\frac{\Lambda(x)dx}{\ln\left(\frac{g_p(t_1)}{g_p(x)}\right)+m(x)-m(t_1)}\right)}{\ln\left(\frac{g_p(t_1)}{g_p(t)}\right) + m(t) - m(t_1)}, \tag{29}$$

which yields (27). Now, since (25) yields $g_p(t_1)e^{-m(t_1)} = g_p(t_2)e^{-m(t_2)}$, it is clear that the right hand side of (29) goes to infinity as $t \to t_2$ so that we can set $H(t_2) = 1$. Also since p is continuous, g_p is continuous and H is continuous in $(t_1, t_2]$.

To see that H is non-decreasing assume for simplicity that p is differentiable. In this case

$$H'(t) = \frac{{g_p}'(t)}{g_p(t)K\exp\left(-\int_{t}^{(t_1+t_2)/2}\frac{\Lambda(x)dx}{\ln\left(\frac{g_p(t_1)}{g_p(x)}\right) + m(x) - m(t_1)}\right)} \geq 0.$$

In general the monotonicity of H can be easily obtained by considering a sequence $(g_p^n)_{n\in\mathbb{N}} \in \mathcal{C}^\infty([0,T])$ of non-decreasing functions such that $g_p^n \to g_p$ uniformly on $[0,T]$. To conclude we note that the conditional distributions defined in (28) are indeed distributions. Since H is continuous and non-decreasing in $(t_1, t_2]$, H_t is also continuous and non-decreasing in $[t, t_2]$. Moreover, since $H(t_2) = 1$, $H_t(t_2) = 1$ and $H_t(t) = 0$. □

Remark 6. *H remains constant on a subset of (t_1, t_2) if and only if g_p remains constant.*

Now we tackle the general case when the assumption on the monotonicity of g_p is dropped. For this we define an auxiliary price scheme over (t_1, t_2)

$$\bar{p}(t) := V + \inf_{\tau \in (t_1, t)} \{e^{\mu(t-\tau)}(p(\tau) - V)\} = V - e^{\mu t} \sup_{\tau \in (t_1, t)} g_p(\tau), \tag{30}$$

and define the distribution, $\bar{H}$, corresponding to the mixing interval (t_1, t_2) as in (27) but using $g_{\bar{p}}$ instead of g_p. Similarly, we define the strategy for $t \in (t_1, t_2)$ as $\bar{H}_t$, the conditional distribution of $\bar{H}$ obtained as in (28).

To see the intuition behind $\bar{p}$, note that when $\mu = 0$, $\bar{p}(t) = \inf_{\tau \in (t_1, t)} p(\tau)$. In general, $\bar{p}$ is the largest function below p such that $g_{\bar{p}}(t) = \sup_{\tau \in (t_1, t)} g_p(\tau)$ is non-decreasing. Indeed, whenever g_p is decreasing, $g_{\bar{p}}$ remains constant, and furthermore, if in an interval $g_{\bar{p}} \neq g_p$, then $g_{\bar{p}}$ is constant in that interval.

Notice that $\bar{H}$ and $\bar{H}_t$ are indeed distributions. To see this observe that $p(t) = \bar{p}(t)$ if and only if $g_p(t) = g_{\bar{p}}(t)$ and therefore the set $A = \{t \in (t_1, t_2) |\ \bar{p}(t) \neq p(t)\}$ is the same set as the set where $g_{\bar{p}}(\cdot)$ remains constant. Thus, by Remark 6, the support of $\bar{H}$ is actually $[t_1, t_2] \setminus A$ and therefore $\bar{H}$ and $\bar{H}_t$ are non-decreasing. Also, by Lemma 4, over a mixing interval the set where $f(\cdot)$ stays constant is the set A.

Remark 7. *The analysis for the mixing interval (t_0, t_*) is analogous, except that in the computation of f we consider that $Q_k(t) = e^{-m(t)} m(t)^k / k!$ since buyers arriving in $[0, t_0]$ are waiting to buy on (t_0, t_*).*

A.3. **Putting the pieces together.** We are now ready to prove Theorem 2. First, recall that the strategies of buyers in the game can be summarized as follows:

(i) Consumers arriving in I buy upon arrival.
(ii) Consumers arriving at time $t \in [0, t_0]$ buy at a random time drawn according to the distribution $\bar{H}$ corresponding to the mixing interval (t_0, t_*) constructed using (27), (30) and Remark 7.
(iii) Consumers arriving at time $t \in (t_0, t_*)$ buy at a random time drawn according to the conditional distribution $\bar{H}_t$,corresponding to distribution $\bar{H}$ of the mixing interval (t_0, t_*).
(iv) Consumers arriving at time $t \in (t_1, t_2)$ of a generic mixing interval buy at a random time drawn according to the conditional distribution $\bar{H}_t$, defined by (28) and (30), which corresponds to the distribution $\bar{H}$ of the mixing interval (t_1, t_2).

In the next lemma we show that the time decomposition of the horizon $[0, T]$ is correct. This implies that for every $t \in [0, T]$ we can associate a strategy as just described.

Lemma 8. *The interval $[0, T]$ can be partitioned into the set I, the interval $[0, t_*)$, and a collection of mixing intervals of the form (t_1, t_2), as defined by equations (21)–(25). Hence, the constructed strategies are well defined for every $t \in [0, T]$.*

Proof. We first prove that the set I is a compact. Indeed, consider a convergent sequence in I, $x_n \to x$ and $\varepsilon > 0$. By continuity of $T(s) := (V - p(s))e^{-(\mu s + m(s))}$ there exists $M > 0$ such that

$$\max_{s \geq x_n} T(s) + \varepsilon \geq \max_{s \geq x} T(s) \quad \text{for all } n \geq M,$$

since $x_n \in I$ we obtain that $T(x_n) + \varepsilon \geq \max_{s \geq x} T(s)$ for all $n \geq M$. Taking $n \to \infty$ we conclude that $x \in \arg\max_{s \geq x} T(s)$, equivalently $x \in I$. Then, I is closed and thus compact.

The compactness of I quickly implies the lemma. Let $t \in [0, T]$ and let $\bar{t} = \min\{s \geq t : s \in I\}$. Clearly, if $t = \bar{t}$, then $t \in I$. So assume $t < \bar{t}$. In this case, if $\bar{t} = t_*$, then $t \in [0, t_*]$. Otherwise it must happen that there exists $s \in I$ such that $s < t$. So letting $t_1 = \max\{s \leq t |\ s \in I\}$ and $t_2 = \bar{t}$, they satisfy (25) and (24), respectively, therefore $t \in (t_1, t_2)$. ☐

Slightly abusing notation let us call H_t the strategy thus defined for a buyer arriving at any time $t \in [0, T]$. We are finally able to prove our main result stating that if all buyers behave accordingly we have a Bayes-Nash equilibrium of the game.

Proof of Theorem 2. First, let us show that if H_t is an equilibrium when p is continuous, then it will also be an equilibrium when p is only continuous over $[0, T)$ and $p(T) = v$. In fact, if all players follow H_t, the utility of any player arriving before T is greater than or equal to $(V - p(T^-))e^{-(\mu T + m(T))}$. Then, since $p(T^-) \leq V - \frac{(V-v)(1-e^{-\int_0^T \lambda(t)dt})}{\int_0^T \lambda(t)dt}$, we conclude that is not profitable to wait and purchase at end of the season.

In order to conclude that H_t is an equilibrium for the game, it only remains to prove that there is no profitable deviation over $[0, T)$. By contradiction, assume that there is a deviation for some player, i.e., there exists $t \in [0, T)$ and $z \geq t$ such that z is outside the support of H_t and $U(z) > U(s)$ for every s in the support of H_t.

If $t \in I$, we have that $H_t(t) = 1$ (buy upon arrival) and by definition of I we obtain $(V - p(t))e^{-(\mu t + m(t))} \geq (V - p(z))e^{-(\mu z + m(z))} \geq U(z)$. The last inequality follows since in our defined strategies every buyer arriving before t will end up buying before t.

Otherwise, if t belongs to a mixing interval of the form $t \in (t_1, t_2)$ we recall that the utility of a buyer arriving at time t equals $U(t_1)$. Now for $z \in (t_1, t_2]$ we have that $U(z) \leq U(t_1)$, where, by Lemma 5, equality holds if $p(z) = \bar{p}(z)$, and strict inequality otherwise since $g_{\bar{p}}(z) > g_p(z)$. If on the contrary $z > t_2$ the situation is analogous to the case $t \in I$ since by definition of t_2, $(V - p(t_2))e^{-(\mu t_2 + m(t_2))} \geq (V - p(z))e^{-(\mu z + m(z))} \geq U(z)$.

Finally we study the case $t \in [0, t_*]$ with $t_* > 0$. If $z \geq t_0$ the situation is analogous to the previous case. If on the contrary $z < t_0$ assume for a contradiction that $U(z) > U(t_0)$. On the other hand, since $t_* > 0$, we have that $U(0) < U(t_*) = U(t_0)$, therefore there exists $\bar{z} < t_0$ such that $U(\bar{z}) = (V - p(\bar{z}))e^{-\mu \bar{z}} = U(t^*) = (V - p(t_*))e^{-(\mu t_* + m(t_*))}$, which contradicts the minimality of t_0. $\square$