Land use planning and optimal subsidies *

L.M. Briceño-Arias[a] and F. Martínez[b]

[a] *Departamento de Matemática, Universidad Técnica Federico Santa María, Santiago, Chile*;
[b] *Instituto Sistemas Complejos de Ingeniería, Universidad de Chile, Santiago, Chile*

(*Received 00 Month 201X; final version received 00 Month 201X*)

Urban planning is a complex problem which includes choosing a social objective for a city, finding the associated optimal allocation of agents and identifying instruments like subsidies to decentralize this allocation as a market equilibrium. We split the problem in two independent steps. First, we find the optimal allocation for a social objective and, second, we derive subsidies that reproduce the optimal allocation as a market equilibrium. This splitting is supported by a fundamental result asserting that the optimal allocation of any social objective can be decentralized by applying feasible subsidies, which can be computed even in the case with location externalities. In the first step, we compute the optimal allocation using an algorithm to solve a convex urban planning problem, which is applicable to a wide class of objective functions. In the second step, we compute optimal subsidies in several political situations for the planner, like budget constraints and limited impact on specific agents, zones, rents and/or utilities. As an example, we simulate a prototype city which aims at improving social inclusion.

Keywords: convex optimization; land use planning problem; location subsidies; urban segregation.

1. Introduction

Megacities face chronic problems like congestion, social segregation, urban sprawl, and high land rents, in addition to crime and the concern about climate change. These can be seen as costs of development, but there are also opportunities for the decision maker to implement policies that reduces these negative impacts. Methods to study how to plan cities have so far concentrated on simulating the long term impacts of project and policies defined as future scenarios. Land use and transport models contribute in this task forecasting the impact of each scenario considered and is fair to say that these models have advanced in the last two decades to become operational and widely used by practitioners, which can be seen in Hunt et al.7 (2005); Preston et al. (2010); Timmermans and Zhang (2009); and Wegener (1994, 1998).

However, the scenario approach leaves the planner with two enormous tasks after a social goal is identified: build wise scenarios that improve the social objective and implement policies to achieve that scenario. This is a complex problem because because the combinatorial of potential scenarios and their associated policies is too large, and

*Corresponding author: fmartine@ing.uchile.cl. This work is supported by the Complex Engineering Systems Institute, ISCI (ICM-FIC: P05-004-F, CONICYT: FB0816), by project FONDECYT 11140360, and by "Programa de financiamiento basal" from the Center for Mathematical Modeling, Universidad de Chile. The authors thanks C. Vigouroux for his help with simulations.

testing each scenario is computationally costly. Consider for example a planner who has already identified the optimal scenario, i.e. has solved the first task, and must define appropriate subsidies as policies to implement the scenario using the market, i.e., to decentralize the policy. In this simple case, in a city with M socioeconomic groups and N zones, the planner must define $N \times M$ subsidies/taxes. For example, in a small city with $M = 10$ and $N = 1.000$ we have 1×10^4 location subsidies to be defined. In sum, the optimal urban planning is not feasible via testing scenarios; there is a need to develop models able to optimize the city by efficiently searching in the combinatorial number of scenarios and subsidies.

The simpler task of modeling integrated land use and transport (LUT) scenarios is already a complex mathematical problem, which researchers have found difficult to solve providing confidence on the results in the presence of location externalities and transport congestion. Some models, e.g. Ma and Lo (2012), solve the equilibrium of the integrated LUT problem, including congestion and without location externalities, but they cannot guarantee the uniqueness of the solution. Briceño-Arias et al. (2008) solved the LUT problem for a unique solution, extended by Bravo et al. (2010) for the case with location externalities. Some approaches addressing the planning problem in the LUT system are Ma and Lo (2012) and Ying (2015). However, none of these models attempt to optimize the city location distribution regarding a social goal. This might be justified by the extra difficulty of adding to the LUT problem the complexity of finding an optimal allocation and the corresponding implementation policies.

The above difficulties inspire us to simplify in this paper the planning and subsidies problem by assuming transportation costs as given. In this context, we consider the case where the planner can apply subsidies/taxes on residents to attain the desired optimal allocation. Then, the urban optimal planning problem can be formulated as follows:

$$\underset{s \in \mathrm{R}^{N \times M}}{\text{minimize}} \;\; \psi(x(s)) \tag{1.1}$$

$$\text{s.t. } x(s) \in \Xi, \tag{1.2}$$

where $s \in \mathrm{R}^{N \times M}$ is the vector of subsidies, $x(s)$ is the vector describing the market equilibrium allocation in the city after applying subsidies s, which is constrained to the set of market clearing allocations Ξ, and ψ is the social objective function to be minimized, e.g., social exclusion. In our approach, this problem is decoupled in two subproblems. First, the *planning problem* defined by

$$\underset{x \in \mathrm{R}^{|C|}}{\text{minimize}} \;\; \psi(x) \tag{1.3}$$

$$\text{s.t. } x \in \Xi, \tag{1.4}$$

whose solution is the optimal allocation x^* and it is independent of the policy instruments the planner may use. Second, the *optimal subsidies problem*, which finds a set of subsidies/taxes that decentralize the policy to attain the optimal allocation via the market equilibrium, i.e., $x(s) = x^*$.

Therefore, in the first step we find an optimal allocation $\boldsymbol{x}^*$, which is the solution to the land use planning problem. We solve this problem for a large class of objective functions including, for instance, social inclusion measures or the utilitarian Benthamite social function (i.e., maximize agents' utilities). Second, we compute the optimal subsidies by zone and agent, using a fundamental result, which asserts that any feasible allocation can be attained decentralized by some set of subsidies. That is, by applying this set of subsidies, agents behave freely in the location market and the resulting equilibrium is optimum.

In Section 2, we formulate the planning problem in terms of its primal and dual problems, where the land-use equilibrium problem is a particular case that complies with the utilitarian social objective. We also prove existence and uniqueness of this optimal primal-dual solution under suitable conditions. In Section 3, we find the set of feasible subsidies that make the optimum allocation $\boldsymbol{x}^*$ to be an equilibrium. Since this set is not a singleton, we then propose several policies to define subsidies complying with additional criteria. In Section 4 we study an application of the approach to the social inclusion problem, i.e., maximize an inclusion index, then we consider cases when the policy maker faces some implementation constraints and we provide numerical simulations.

2. The planning problem

In this section we formulate the planning problem as a primal-dual formulation for a wide range of objective functions; the market equilibrium problem is represented by one of them, making this problem a special case of the planning problem. Let C be the set of types of households and suppose that one firm controls the real estate supply. For every zone $i \in N$, let $S_i \in \,]0,+\infty[$ be the supply in the zone i and, for every $h \in C$, let $H_h \in \,]0,+\infty[$ be the demand of the households type h in the land use market. For every $h \in C$ and $i \in N$, we denote by x_{hi} the number of households type h localized in i and we set $z_{hi} \in \mathrm{R}$ be the utility perceived by the household h on the amenities at zone i, including real estate attributes and accessibility to the household activities. These utilities are assumed to be constant and known. Hence, transportation costs and other location attributes are assumed to be exogenous, except for location externalities which are studied at the end of Section 3. Additionally we assume the market clearing condition $T = \sum_{i\in N} S_i = \sum_{h\in C} H_h$, i.e., we suppose that the number of households demanding for a zone coincides with the number of houses available.

Let us denote as $\mathscr{C}$ the class of functions $\psi\colon \mathrm{R} \times [0,+\infty[\, \to \,]-\infty,+\infty]$ such that

$$(\forall z \in \mathrm{R}) \quad \begin{cases} \psi(z,\cdot) \text{ is strictly convex,} \\ \operatorname{dom}\psi(z,\cdot) = [0,a[\,, \text{ for some } a \in \,]0,+\infty]\,, \\ \lim_{x\to+\infty} \psi(z,x) = +\infty. \end{cases} \tag{2.1}$$

This class defines the set of objective functions that we consider for solving the land use planning problem defined as follows.

Problem 2.1 (Land use planning - primal problem)**.** *For every $h \in C$ and $i \in N$, let $z_{hi} \in \mathrm{R}$, let $\psi_{hi} \in \mathscr{C}$ with domain $[0, a_{hi}[$ for some $a_{hi} \in]0,+\infty]$, let*

$$\Xi = \Big\{ \boldsymbol{x} \in \mathrm{R}^{|C|\times|N|} \;\Big|\; (\forall i \in N) \quad \sum_{h\in C} x_{hi} = S_i \quad \textit{and} \quad (\forall h \in C) \quad \sum_{i\in N} x_{hi} = H_h \Big\}, \tag{2.2}$$

and suppose that

$$\Xi \cap \bigtimes_{h\in C}\bigtimes_{i\in N} [0, a_{hi}[\,\neq \varnothing. \tag{2.3}$$

The problem is to

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}} \quad \sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},x_{hi}). \tag{2.4}$$

The objective function of this problem belongs to the class $\mathscr{C}$, that is, it is strictly convex and coercive. Additionally we allow functions $(\psi_{hi})_{h\in C,\,i\in N}$ to have as domain all the positive real numbers ($a_{hi}=+\infty$) or, if necessary, have a restricted domain ($a_{hi}<+\infty$) if the function is unbounded.

Proposition 2.2 (Dual problem)**.** *Under the assumptions of Problem 2.1, the dual problem associated to* (2.4) *is*

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in\mathrm{R}^{|C|+|N|}}{\text{minimize}} \quad \boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r}) := \sum_{h\in C}H_h b_h + \sum_{i\in N}S_i r_i + \sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},-b_h-r_i), \tag{2.5}$$

where, for every $h\in C$ and $i\in N$,

$$\varphi_{hi}\colon (z_{hi},u)\mapsto \psi_{hi}(z_{hi},\cdot)^*(u) := \sup_{x\in[0,a_{hi}[}(\langle x\mid u\rangle - \psi_{hi}(z_{hi},x)). \tag{2.6}$$

Proof. See Appendix. □

The function φ_{hi} defined in (2.6) is known as the Fenchel conjugate of $\psi_{hi}(z_{hi},\cdot)$, which is denoted by $\psi_{hi}(z_{hi},\cdot)^*$ and is also convex.

Note that, for every $(\boldsymbol{b},\boldsymbol{r})\in\mathrm{R}^{|C|+|N|}$ and $\alpha\in\mathrm{R}$, $\boldsymbol{\Phi}(\boldsymbol{b}+\alpha,\boldsymbol{r}-\alpha)=\boldsymbol{\Phi}(\boldsymbol{b},\boldsymbol{r})$, where $b+\alpha=(b_1+\alpha,\ldots,b_{|C|}+\alpha)$ and $r-\alpha=(r_1-\alpha,\ldots,r_{|N|}-\alpha)$, which follows from the market clearing assumption $\sum_{h\in C}H_h=\sum_{i\in N}S_i$. Hence, for having uniqueness of the solution, we have to consider some additional constraints in the dual problem.

Proposition 2.3 (Uniqueness of the dual solution)**.** *Problem 2.1 has a unique solution. In addition, if the dual problem considers one of the following constraints D_1 to D_4 for some $\eta\in\mathrm{R}$, then the dual problem* (2.5) *has also a unique solution.*

(i) $\boldsymbol{b}\in D_1=\big\{\boldsymbol{b}\in\mathrm{R}^{|C|}\;\big|\;b_1=\eta\big\}$.
(ii) $\boldsymbol{b}\in D_2=\big\{\boldsymbol{b}\in\mathrm{R}^{|C|}\;\big|\;\frac{1}{|C|}\sum_{h\in C}b_h=\eta\big\}$.
(iii) $\boldsymbol{r}\in D_3=\big\{\boldsymbol{r}\in\mathrm{R}^{|N|}\;\big|\;r_1=\eta\big\}$.
(iv) $\boldsymbol{r}\in D_4=\big\{\boldsymbol{r}\in\mathrm{R}^{|N|}\;\big|\;\frac{1}{|N|}\sum_{i\in N}r_i=\eta\big\}$.

Moreover, primal and dual solutions are related via $\overline{x}_{hi}=\big(\varphi_{hi}(z_{hi},\cdot)\big)'(\overline{b}_h+\overline{r}_i)$, for every $h\in C$ and $i\in N$, where $(\varphi_{hi}(z_{hi},\cdot))_{h\in C,i\in N}$ are defined in (2.6)*. In the rest of this paper we assume $b_1=0$ in order to guarantee uniqueness of the primal-dual solution.*

Proof. See Appendix. □

The variety of objectives functions comprised in the planning problem (2.5) can be solved using the algorithm described in Appendix, where the convergence to the solution is proved.

2.1 *Land use market equilibrium*

The equilibrium in the land-use market can be seen as a particular case of Problem 2.1. Indeed, when, for every $h\in C$ and $i\in N$, $\psi_{hi}\colon(z,x)\mapsto -xz+x(\ln x-1)/\mu\in\mathscr{C}$, for some $\mu \in]0, +\infty[$, Problem 2.1 becomes

$$\underset{\boldsymbol{x}\in\Xi}{\text{minimize}} \sum_{h\in C}\sum_{i\in N} -x_{hi}z_{hi} + \frac{1}{\mu}x_{hi}(\ln x_{hi} - 1), \tag{2.7}$$

which is the well known entropy maximizing problem, formulated in Briceño-Arias et al. (2008) as an equivalent bid-rent equilibrium problem. Moreover, in this case it is easy to verify that, for every $(h, i) \in C \times N$, $\varphi_{hi}(z, u) = \psi_{hi}(z_{hi}, \cdot)^*(u) = e^{\mu(z+u)}/\mu$. and, hence, (2.5) becomes

$$\underset{(\boldsymbol{b},\boldsymbol{r})\in \mathrm{R}^{|C|+|N|}}{\text{minimize}} \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i + \frac{1}{\mu}\sum_{h\in C}\sum_{i\in N} e^{\mu(z_{hi}-b_h-r_i)}, \tag{2.8}$$

which is the dual problem associated to (2.7) (see Briceño-Arias et al. 2008). The first order optimality conditions of (2.8) are

$$(\forall h \in C)(\forall i \in N) \quad \begin{cases} \sum_{i\in N} e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))} = H_h \\ \sum_{h\in C} e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))} = S_i \end{cases} \tag{2.9}$$

and we deduce that the solution to the primal problem is $x_{hi}(\boldsymbol{z}) = e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))}$, where we denote by $\boldsymbol{x}(\boldsymbol{z}) = (x_{hi}(\boldsymbol{z}))_{h,i}$, $\boldsymbol{b}(\boldsymbol{z}) = (b_h(\boldsymbol{z}))_{h\in C}$ and $\boldsymbol{r}(\boldsymbol{z}) = (r_i(\boldsymbol{z}))_{i\in N}$ the unique primal and dual solutions, respectively.

In the primal and dual problems (2.7) and (2.8), $(\boldsymbol{b}(\boldsymbol{z}), \boldsymbol{r}(\boldsymbol{z})) \in \mathrm{R}^{|C|+|N|}$ represent the Lagrange multipliers of the constraints (2.2), for total households and total supply, respectively. These multipliers have economic interpretation considering the equivalent model derived from the random bidding framework by Ellickson (1981) and Martínez (1992), where $\boldsymbol{b}(\boldsymbol{z})$ and $\boldsymbol{r}(\boldsymbol{z})$ represent the vectors of utilities and rents that clears the market. Indeed, from (2.9) we can isolate $b_h(\boldsymbol{z})$ from the first equation and $r_i(\boldsymbol{z})$ from the second, obtaining

$$(\forall h \in C)(\forall i \in N) \quad \begin{cases} b_h(\boldsymbol{z}) &= \frac{1}{\mu}\ln\left(\frac{1}{H_h}\sum_{i\in N} e^{\mu(z_{hi}-r_i(\boldsymbol{z}))}\right) \\ r_i(\boldsymbol{z}) &= \frac{1}{\mu}\ln\left(\frac{1}{S_i}\sum_{h\in C} e^{\mu(z_{hi}-b_h(\boldsymbol{z}))}\right) \end{cases} \tag{2.10}$$

and, by replacing $r_i(\boldsymbol{z})$ from (2.10) in $x_{hi}(\boldsymbol{z}) = e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))}$ we obtain

$$(\forall h \in C)(\forall i \in N) \quad x_{hi}(\boldsymbol{z}) = S_i\frac{e^{\mu(z_{hi}-b_h(\boldsymbol{z}))}}{\sum_{g\in C} e^{\mu(z_{gi}-b_g(\boldsymbol{z}))}}. \tag{2.11}$$

In the random bidding framework (Ellickson (1981); Martínez (1992)), this expression represents the expected number of agents of type h located at zone i with S_i locations; r_i is the expected rent obtained in the auction of i; $z_{hi} - b_h$ is the expected value of agent h's bid for zone i, with b_h the utility level that the agent obtains at equilibrium. Similarly, by replacing $b_h(\boldsymbol{z})$ from (2.10) in $x_{hi}(\boldsymbol{z}) = e^{\mu(z_{hi}-b_h(\boldsymbol{z})-r_i(\boldsymbol{z}))}$ we obtain

$$(\forall h \in C)(\forall i \in N) \quad x_{hi}(\boldsymbol{z}) = H_h\frac{e^{\mu(z_{hi}-r_i(\boldsymbol{z}))}}{\sum_{j\in N} e^{\mu(z_{hj}-r_j(\boldsymbol{z}))}}, \tag{2.12}$$

which is the logit choice model involving the household h's surplus, proposed in Anas

(1982). This shows the equivalence between the bidding and choice models under conditions (2.9).

It is worth remarking that, for every utility of attributes $\tilde{\boldsymbol{z}}$, in particular, for $\tilde{\boldsymbol{z}} = \boldsymbol{z} + \boldsymbol{s}$ where $\boldsymbol{s}$ can be subsidies, the market equilibrium condition (2.10) establishes a link between rents $\boldsymbol{r}(\tilde{\boldsymbol{z}})$ and utilities $\boldsymbol{b}(\tilde{\boldsymbol{z}})$ via $b_h(\tilde{\boldsymbol{z}}) = f_h(\tilde{\boldsymbol{z}}, \boldsymbol{r}(\tilde{\boldsymbol{z}}))$ and $r_i(\tilde{\boldsymbol{z}}) = g_i(\tilde{\boldsymbol{z}}, \boldsymbol{b}(\tilde{\boldsymbol{z}}))$, where

$$f_h(\tilde{\boldsymbol{z}}, \boldsymbol{r}) = \frac{1}{\mu} \ln \left(\frac{1}{H_h} \sum_{i \in N} e^{\mu(\tilde{z}_{hi} - r_i)} \right) \quad \text{and} \quad g_i(\tilde{\boldsymbol{z}}, \boldsymbol{b}) = \frac{1}{\mu} \ln \left(\frac{1}{S_i} \sum_{h \in C} e^{\mu(\tilde{z}_{hi} - b_h)} \right). \quad (2.13)$$

Moreover, combining both equations we derive the fixed point system $\boldsymbol{b}(\tilde{\boldsymbol{z}}) = f(\tilde{\boldsymbol{z}}, g(\tilde{\boldsymbol{z}}, \boldsymbol{b}(\tilde{\boldsymbol{z}})))$, yielding the equilibrium value for $\boldsymbol{b}(\tilde{\boldsymbol{z}})$ (Martínez and Henríquez 2007).

3. Optimal subsidies

We define optimal subsidies as a set of values (subsidies and taxes) which satisfy two objectives: first, they are feasible in the sense that they lead to the planned allocation $\boldsymbol{x}^* = (x^*_{hi})_{h \in C, i \in N}$ and, second, in the case when the feasible set is too large, they comply with an additional optimizing criteria.

3.1 *Feasible subsidies problem*

In this problem, we look for feasible subsidies/taxes $\boldsymbol{s} = (s_{hi})_{h \in C, i \in N}$, which should be given/charged to each type of household in each zone such that the agents behavior is altered in order to attain a planned allocation $\boldsymbol{x}^* = (x^*_{hi})_{h \in C, i \in N}$ as an equilibrium. We first study the case without location externalities, where a vector of preferences $\boldsymbol{z} = (z_{hi})_{h \in C, i \in N}$ is exogenous, and then we extend the analysis for the general case in Section 3.1.2, where $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x}^*)$.

3.1.1 The case without externalities

To formulate the problem, recall that, for every exogenous vector $\boldsymbol{z} = (z_{hi})_{h \in C, i \in N}$, $\boldsymbol{x}(\boldsymbol{z}) = (x_{hi}(\boldsymbol{z}))_{h \in C, i \in N}$, given by $x_{hi}(\boldsymbol{z}) = e^{\mu(z_{hi} - b_h(\boldsymbol{z}) - r_i(\boldsymbol{z}))}$, is the corresponding unique solution to the primal unsubsidized equilibrium problem (2.7), where $(b_h(\boldsymbol{z}))_{h \in C}$ and $(r_i(\boldsymbol{z}))_{i \in N}$ are the unique solution to the dual market equilibrium problem (2.8) under the additional constraint $b_1 = 0$. These dual solutions are related via the equation (2.10).

Problem 3.1. *(Feasible Subsidies Problem) Given a desired allocation* $\boldsymbol{x}^* = (x^*_{hi})_{h \in C, i \in N}$*, the problem is to find* $\boldsymbol{s} = (s_{hi})_{h \in C, i \in N} \in \mathrm{R}^{|C| \times |N|}$ *such that* $\boldsymbol{x}(\boldsymbol{z} + \boldsymbol{s}) = \boldsymbol{x}^*$.

In (Águila 2006, P. 44) this problem is addressed in the context of variable supply and we reformulate it in the following proposition proved in (Águila 2006, A.25).

Proposition 3.2. *Let* $(x^*_{hi})_{h \in C, i \in N} \in \Xi$ *be a desired allocation and let* $\boldsymbol{\beta} = (\beta_h)_{h \in C} \in \mathrm{R}^{|C|}$ *with* $\beta_1 = 0$ *be an arbitrary vector. The subsidy* $\boldsymbol{t}(\boldsymbol{\beta})$ *obtained from the fixed point equation*

$$(\forall h \in C)(\forall i \in N) \quad t_{hi} = \frac{1}{\mu} \ln x^*_{hi} + \beta_h + g_i(\boldsymbol{z} + \boldsymbol{t}, \boldsymbol{\beta}) - z_{hi}, \quad (3.1)$$

is a solution to Problem 3.1, where g_i *is defined in* (2.13)*. Moreover, for every* $h \in C$*, we have* $b_h(\boldsymbol{z} + \boldsymbol{t}(\boldsymbol{\beta})) = \beta_h$*. Symmetrically, given* $\boldsymbol{\rho} = (\rho_i)_{i \in N} \in \mathrm{R}^N$ *with* $\rho_1 = 0$*, the* *subsidy $\widetilde{\boldsymbol{t}}(\boldsymbol{\rho})$ defined by the fixed point equation*

$$(\forall h \in C)(\forall i \in N) \quad \widetilde{t}_{hi} = \frac{1}{\mu} \ln x^*_{hi} + f_h(\boldsymbol{z} + \widetilde{\boldsymbol{t}}, \boldsymbol{\beta}) + \rho_i - z_{hi}, \tag{3.2}$$

is also a solution to Problem 3.1, where f_h is defined in (2.13). Moreover, for every $i \in N$, we have $r_i(\boldsymbol{z} + \widetilde{\boldsymbol{t}}(\boldsymbol{\rho})) = \rho_i$.

We observe that, for every $\boldsymbol{\beta} \in \mathrm{R}^{|C|}$ we obtain a unique subsidy vector $\boldsymbol{t}(\boldsymbol{\beta}) \in \mathrm{R}^{|N|\times|C|}$, which defines a bijection between $\mathrm{R}^{|C|}$ and the set of solution subsidies $\boldsymbol{t}$ computed via (3.1). In the following theorem, Proposition 3.2 is extended allowing larger flexibility for the planner in setting subsidies leading to the planned allocation $\boldsymbol{x}^*$; more precisely, we define a bijection between $\mathrm{R}^{|C|+|N|}$ and the set of solutions proposed below. Additionally, the proposed subsidies can be computed explicitly, avoiding the fixed point procedure in (3.1).

Theorem 3.3. *Let $(x^*_{hi})_{h\in C, i\in N} \in \Xi$ be a desired allocation, let $\boldsymbol{\beta} = (\beta_h)_{h\in C} \in \mathrm{R}^{|C|}$ with $\beta_1 = 0$ and $\boldsymbol{\rho} = (\rho_i)_{i\in N} \in \mathrm{R}^{|N|}$ be arbitrary vectors. The following subsidy*

$$(\forall h \in C)(\forall i \in N) \quad s_{hi}(\beta_h, \rho_i) = \frac{1}{\mu} \ln x^*_{hi} + \beta_h + \rho_i - z_{hi}, \tag{3.3}$$

is a solution to Problem 3.1. Moreover, for every $h \in C$ and $i \in N$, we have $b_h(\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \beta_h$ and $r_i(\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \rho_i$.

Proof. Let $\boldsymbol{\beta} = (\beta_h)_{h\in C} \in \mathrm{R}^{|C|}$ and $\boldsymbol{\rho} = (\rho_i)_{i\in N} \in \mathrm{R}^{|N|}$ be any vector satisfying $\beta_1 = 0$. We have from (2.9) that $(b_h(\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})))_{h\in C}$ and $(r_i(\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})))_{i\in N}$ are the unique solution to the non-linear system

$$(\forall h \in C)(\forall i \in N) \quad \begin{cases} \sum_{i\in N} e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))} = H_h \\ \sum_{h\in C} e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))} = S_i. \end{cases} \tag{3.4}$$

It follows from (3.3) that (3.4) is equivalent to

$$(\forall h \in C)(\forall i \in N) \quad \begin{cases} \sum_{i\in N} x^*_{hi} e^{\mu(\beta_h-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))+\rho_i-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))} = H_h \\ \sum_{h\in C} x^*_{hi} e^{\mu(\beta_h-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))+\rho_i-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))} = S_i. \end{cases} \tag{3.5}$$

Hence, since $(x^*_{hi})_{h\in C, i\in N} \in \Xi$ we have that the unique solution to the system under the constraint $\beta_1 = 0$ is, for every $h \in C$ and $i \in N$, $b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \beta_h$ and $r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \rho_i$. Moreover, we deduce from (3.3) that, for every $h \in C$ and $i \in N$,

$$x_{hi}(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-b_h(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho}))-r_i(\boldsymbol{z}+\boldsymbol{s}(\boldsymbol{\beta},\boldsymbol{\rho})))} = e^{\mu(z_{hi}+s_{hi}(\beta_h,\rho_i)-\beta_h-\rho_i)} = x^*_{hi}, \tag{3.6}$$

which yields the result. This means that, by considering the modified vector of utilities $\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})$ instead of $\boldsymbol{z}$, the land use equilibrium allocation coincides with the desired allocation $\boldsymbol{x}^*$ and the equilibrium utilities and rents coincide with the arbitrary chosen ones. □

Remark 3.4. *Note that, for arbitrary vectors $(\beta_h)_{h\in C}$ and $(\rho_i)_{i\in N}$, Theorem 3.3 ensures the existence of a subsidy $\boldsymbol{s} = (s_{hi})_{h\in C, i\in N}$ given by (3.3), such that the resulting mutually dependent rents and utilities at the new equilibrium are $(\beta_h)_{h\in C}$ and $(\rho_i)_{i\in N}$. For the planner, this means that she can choose a priori any utilities and rents $(\beta_h)_{h\in C}$ and $(\rho_i)_{i\in N}$, which along with the associated subsidies given by (3.3), yield the desired* *allocation $\boldsymbol{x}^*$. This flexibility can be used by the planner to define policy instruments as we discuss below.*

3.1.2 The case with location externalities

Location externalities are associated with the perception of agents regarding neighborhood quality, which depends on the location of other agents. Then, $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{x})$ and the equilibrium conditions $x_{hi} = e^{\mu(z_{hi}(\boldsymbol{x})-b_h(\boldsymbol{z}(\boldsymbol{x}))-r_i(\boldsymbol{z}(\boldsymbol{x})))}$ defines a fixed point on $\boldsymbol{x}$ and it is no longer solution to (2.7). Therefore, given an objective allocation $\boldsymbol{x}^*$ and the endogeneity of location externalities, we aim at finding optimal subsidies $\boldsymbol{s}$ satisfying the condition $x^*_{hi} = e^{\mu(z_{hi}(\boldsymbol{x}^*)+s_{hi}-b_h(\boldsymbol{z}(\boldsymbol{x}^*)+\boldsymbol{s})-r_i(\boldsymbol{z}(\boldsymbol{x}^*)+\boldsymbol{s}))}$.

Following Theorem 3.3, by using $\boldsymbol{z}(\boldsymbol{x}^*)$ instead of $\boldsymbol{z}$ in (3.3) we obtain

$$(\forall h \in C)(\forall i \in N) \quad s_{hi}(\beta_h, \rho_i) = \frac{1}{\mu}\ln x^*_{hi} + \beta_h + \rho_i - z_{hi}(\boldsymbol{x}^*) \tag{3.7}$$

and replacing this subsidies in (3.5) and (3.6), we deduce

$$(\forall h \in C) \quad b_h(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \beta_h \tag{3.8}$$

$$(\forall i \in N) \quad r_i(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = \rho_i \tag{3.9}$$

$$(\forall (h,i) \in C \times N) \quad x_{hi}(\boldsymbol{z}(\boldsymbol{x}^*) + \boldsymbol{s}(\boldsymbol{\beta}, \boldsymbol{\rho})) = x^*_{hi}. \tag{3.10}$$

From these equations, we deduce that subsidies depend only on the exogenous $\boldsymbol{x}^*$ and also that (3.10) is exactly the equilibrium condition in the case with externalities. Additionally, note that (3.10) does not represent a fixed point in $\boldsymbol{x}^*$, since $\boldsymbol{x}^*$ is exogenous; in fact this equation holds because of the particular choice of subsidies. Therefore, we conclude that the analysis in this case is exactly the same as in the case without externalities.

3.2 *Optimal subsidies problem*

Because the set of feasible subsidies is large, as discussed in Remark 3.4, the planner has a flexibility to choose infinite combinations of rent and utility vectors. Therefore we now propose some examples of policy instruments to help in this task.

Example 3.5 (Maintain rents and utilities as without subsidies)**.** *In this case the policy is to maintain the rents and utilities associated to the market equilibrium without subsidies. For this purpose, the planner should set $\beta_h = b_h(\boldsymbol{z})$ and $\rho_i = r_i(\boldsymbol{z})$, for every $h \in C$ and $i \in N$, which are the utilities and rents obtained by solving the equilibrium problem* (2.8) *under the constraint $\beta_1 = 0$ without subsidies. Then, Theorem 3.3 asserts that the subsidies given by*

$$(\forall h \in C)(\forall i \in N) \quad s_{hi} = \frac{1}{\mu}\ln x^*_{hi} + b_h(\boldsymbol{z}) + r_i(\boldsymbol{z}) - z_{hi}, \tag{3.11}$$

solve Problem 3.1 and comply with the criterion of this example.

Example 3.6 (Minimize resource transferences under social disturbance constraints)**.** *The previous case may yield politically unacceptable high levels of subsidies/taxes. We define the social disturbance as $SD(\boldsymbol{\beta}, \boldsymbol{\rho}) = \sum_{h\in C}(\beta_h - b_h(\boldsymbol{z}))^2 + \sum_{i\in N}(\rho_i - r_i(\boldsymbol{z}))^2$, which represents the deviation of utilities and rents with respect to their values at the unsubsidized equilibrium. In this example, we relax the previous criterion allowing $\boldsymbol{\beta}$ and $\boldsymbol{\rho}$ that comply with a limited social disturbance, in order to reduce subsidies/taxes. By* *considering a gap $\varepsilon > 0$, we propose to*

$$\min_{\boldsymbol{\beta},\boldsymbol{\rho}} \sum_{h\in C}\sum_{i\in N} s_{hi}^2(\beta_h,\rho_i)$$
$$s.t.\ SD(\boldsymbol{\beta},\boldsymbol{\rho}) \leq \varepsilon, \tag{3.12}$$

which yields a unique vector $(\boldsymbol{\beta}^*,\boldsymbol{\rho}^*)$. Then, compute the subsidies

$$(\forall h\in C)(\forall i\in N) \quad s_{hi} = \frac{1}{\mu}\ln x_{hi}^* + \beta_h^* + \rho_i^* - z_{hi}. \tag{3.13}$$

Example 3.7 (Minimize social disturbance under budget constraints)**.** *Now the planner aims to minimize the social disturbance generated by the subsidies policies $s(\boldsymbol{\beta},\boldsymbol{\rho}) = (s_{hi}(\beta_h,\rho_i))_{h\in C,\, i\in N}$ under individuals' and the system budget constraints. Let, for every $h \in C$, $I_h \in [0,+\infty[$, and let $I \in\,]0,+\infty[$. The problem is to*

$$\begin{aligned} &\underset{\boldsymbol{\beta},\boldsymbol{\rho}}{\text{minimize}} \ \ SD(\boldsymbol{\beta},\boldsymbol{\rho}) \\ s.t. \quad & -\sum_{i\in N} s_{hi}(\beta_h,\rho_i) \leq I_h, \quad \forall h\in C \\ & \sum_{h\in C}\sum_{i\in N} s_{hi}(\beta_h,\rho_i) \leq I. \end{aligned} \tag{3.14}$$

This problem includes budgetary constraints for each type of household and for the planner, where, for every $h \in C$, I_h is the representative income of households type h, and I represents the maximum amount of money that the planner wish to use in the subsidies policies. Given a solution $(\boldsymbol{\beta}^*,\boldsymbol{\rho}^*)$ to (3.14)*, subsidies are computed via* (3.13)*. Note that, since it is always possible to find vectors $\boldsymbol{\beta}$ and $\boldsymbol{\rho}$ such that $s_{hi}(\beta_h,\rho_i) = 0$ for every $h \in C$ and $i \in I$, problem* (3.14) *is always feasible.*

Example 3.8 (Constraints on subsidies)**.** *In this example we describe some policies allowing the policy maker to consider some external constraints to set subsidies. In each case we identify a choice of vectors $(\beta_h)_{h\in C}$ and $(\rho_i)_{i\in N}$ from which we compute subsidies from* (3.3) *satisfying the desired constraint. For this purpose, instead of $\beta_1 = 0$, we consider conditions (i)-(iv) in Proposition 2.3 for achieving uniqueness of the dual variables to allow imposing the specific subsidy constraints. The following policy instruments are considered:*

(i) Agent type 1 is not affected*: This means that, for every $i \in N$, $s_{1i}(\beta_1,\rho_i) = 0$ and $\beta_1 = b_1(\boldsymbol{z})$. For this purpose, it is enough to choose $(\beta_h)_{h\in C} \in D_1 = \big\{(\beta_h)_{h\in C} \in \mathrm{R}^{|C|} \bigm| \beta_1 = b_1(\boldsymbol{z})\big\}$, and set*

$$(\forall i\in N) \quad \rho_i = z_{1i} - \frac{1}{\mu}\ln x_{1i}^* - b_1(\boldsymbol{z}). \tag{3.15}$$

The planner still has the flexibility to choose $\beta_2,\ldots,\beta_{|C|}$.

(ii) Zone 1 is not affected*: This means that, for every $h \in C$, $s_{h1}(\beta_h,\rho_1) = 0$ and $\rho_1 = r_1(\boldsymbol{z})$. For this purpose, it is enough to choose $(\rho_i)_{i\in N} \in D_3 =$*

$\{(\rho_i)_{i\in N} \in \mathrm{R}^{|N|} \mid \rho_1 = r_1(\boldsymbol{z})\}$, *and set*

$$(\forall h \in C) \quad \beta_h = z_{h1} - \frac{1}{\mu} \ln x^*_{h1} - r_1(\boldsymbol{z}). \tag{3.16}$$

The planner still has the flexibility to choose $\rho_2, \dots, \rho_{|N|}$.

(iii) Self funded policy by household type*: This means that, for every* $h \in C$, $\sum_{i\in N} s_{hi}(\beta_h, \rho_i) = 0$. *For this purpose, it is enough to choose* $(\rho_i)_{i\in N} \in D_4 = \{(\rho_i)_{i\in N} \in \mathrm{R}^{|N|} \mid \frac{1}{|N|}\sum_{i\in N} \rho_i = \eta\}$, *and set*

$$(\forall h \in C) \quad \beta_h = \frac{1}{|N|} \sum_{i\in N} \Big(z_{hi} - \frac{1}{\mu} \ln x^*_{hi} \Big) - \eta. \tag{3.17}$$

The planner still has the flexibility to choose the reference level of rents and utilities $\eta \in \mathrm{R}$.

(iv) Self funded policy by zone*: This means that, for every* $i \in N$, $\sum_{h\in C} s_{hi}(\beta_h, \rho_i) = 0$. *For this purpose, it is enough to choose* $(\beta_h)_{h\in C} \in D_2 = \{(\beta_h)_{h\in C} \in \mathrm{R}^{|C|} \mid \frac{1}{|C|}\sum_{h\in C} \beta_h = \eta\}$, *and set*

$$(\forall i \in N) \quad \rho_i = \frac{1}{|C|} \sum_{h\in C} \Big(z_{hi} - \frac{1}{\mu} \ln x^*_{hi} \Big) - \eta. \tag{3.18}$$

The planner still has the flexibility to choose the reference level of rents and utilities $\eta \in \mathrm{R}$.

4. Application to the social inclusion problem

In this section, we present the application of land use planning problem and the associated optimal subsidies problem, to the classical example of a society concerned about social inclusion. For this purpose, we consider the case where the policy maker seeks minimizing a measure of spatial socioeconomic heterogeneity along with maximizing agents' utilities. We first analyze the formulation of a social inclusion problem and then we discuss and compute optimal subsidies.

4.1 *Analysis of the problem*

Consider the definitions and notations introduced in Section 2 and, for an allocation $\boldsymbol{x} = (x_{hi})_{h\in C, i\in N} \in \Xi$, consider the following index average

$$I_i(\boldsymbol{x}) := \frac{\sum_{h\in C} x_{hi} I_h}{S_i}, \tag{4.1}$$

where, for every $h \in C$, $I_h \in\,]0, +\infty[$ is a socioeconomic index of households type h, e.g., the household's income. Then we define the following measure for the spatial socioeconomic heterogeneity

$$SE(\boldsymbol{x}) = \sum_{i\in N} \Big(I_i(\boldsymbol{x}) - \overline{I} \Big)^2, \tag{4.2}$$

which is the deviation of the index average in each zone with respect to the city average $\overline{I} = \sum_{h\in C} H_h I_h/T$. Instead of using this measure, which is not separable, in the following proposition we provide a related measure which is separable and convex as the objective function in (2.4).

Proposition 4.1. *Let $\boldsymbol{x} = (x_{hi})_{h\in C, i\in N} \in \Xi$ and define the zone segregation level and the aggregated segregation level by*

$$(\forall i \in N) \quad SL_i(\boldsymbol{x}) = \sum_{h\in C} I_h (x_{hi}/S_i - H_h/T)^2 \quad \text{and} \quad SL(\boldsymbol{x}) = \sum_{i\in N} SL_i(\boldsymbol{x}), \tag{4.3}$$

respectively. Then, $0 \leq SE(\boldsymbol{x}) \leq (\sum_{h\in C} I_h) SL(\boldsymbol{x})$ and the unique minimizer of SL, $\boldsymbol{x}_{SL} = \big((S_i H_h)/T\big)_{h\in C, i\in N}$, coincides with a minimizer of SE.

Proof. Since the SL is strictly convex and coercive it has a unique minimizer $\boldsymbol{x}_{SL} \in \Xi$. Let $\boldsymbol{x}^* = \big((S_i H_h)/T\big)_{h\in C, i\in N}$. Since $SL(\boldsymbol{x}^*) = 0$ and SL is a positive function, it is clear that $\boldsymbol{x}_{SL} = \boldsymbol{x}^*$. Moreover, let $\boldsymbol{x} = (x_{hi})_{h\in C, i\in N} \in \Xi$. It follows from (4.2), (4.3), and Bauschke and Combettes (2011, Lemma 2.13(ii)) that

$$0 \leq SE(\boldsymbol{x}) = \sum_{i\in N} \Big(\sum_{h\in C} I_h (x_{hi}/S_i - H_h/T) \Big)^2 \leq \Big(\sum_{h\in C} I_h \Big) SL(\boldsymbol{x}) \tag{4.4}$$

and, hence, $SE(\boldsymbol{x}_{SL}) = 0$, which yields the result. □

Then, the problem under consideration in this section is to find an allocation which minimizes the aggregated segregation level and, simultaneously, maximizes the total utility. More precisely,

$$\underset{\boldsymbol{x}\in \Xi\cap \mathbb{R}_+^{|C|\times|N|}}{\text{minimize}} \quad -\sum_{h\in C}\sum_{i\in N} x_{hi} z_{hi} + \frac{1}{\alpha} \sum_{h\in C}\sum_{i\in N} I_h (x_{hi}/S_i - H_h/T)^2, \tag{4.5}$$

where Ξ is defined in (2.2) (market clearing) and $\alpha > 0$. This parameter is a measure of the relative importance of the utility of households compared with segregation objective, i.e. the higher is α, the higher is the importance of the utility compared with the segregation for the planner.

Problem (4.5) is a particular case of Problem 2.1 when, for every $(h,i) \in C \times N$, $\psi_{hi}(z_{hi}, \cdot) \colon x \mapsto -x z_{hi} + I_h (x/S_i - H_h/T)^2/\alpha$. Note that functions $(\psi_{hi})_{h\in C, i\in N}$ are in $\mathscr{C}$ and, therefore, Proposition 2.3 asserts that (4.5) has a unique primal solution $\boldsymbol{x}^*_\alpha(\boldsymbol{z})$, which we call *inclusion optimum*, and a unique dual solution for the pair utility and rent $(\boldsymbol{\beta}^*_\alpha(\boldsymbol{z}), \boldsymbol{\rho}^*_\alpha(\boldsymbol{z}))$ under the additional condition $\beta_{1,\alpha} = 0$.

We expect from (4.5) that the segregation level at the optimum is increasing with α. The next proposition asserts that this relationship is indeed quadratic.

Proposition 4.2. *If $\alpha > 0$ is small enough, then the dual solution does not depend on α, i.e., $(\boldsymbol{\beta}^*_\alpha(\boldsymbol{z}), \boldsymbol{\rho}^*_\alpha(\boldsymbol{z})) = (\boldsymbol{\beta}^*(\boldsymbol{z}), \boldsymbol{\rho}^*(\boldsymbol{z}))$, and the segregation level defined in* (4.3) *at the* *inclusion optimum is*

$$SL(\boldsymbol{x}_\alpha^*(\boldsymbol{z})) = \frac{\alpha^2}{4} \sum_{h\in C}\sum_{i\in N} \frac{S_i^2}{I_h}(z_{hi} - \beta_h^*(\boldsymbol{z}) - \rho_i^*(\boldsymbol{z})). \tag{4.6}$$

Proof. Suppose that $\boldsymbol{x}_\alpha^*(\boldsymbol{z})$ is strictly feasible, i.e., that, for every $h \in C$ and $i \in N$, $x_{hi,\alpha}^*(\boldsymbol{z}) > 0$. Then, the first order conditions of (4.5) yield

$$(\forall h \in C)(\forall i \in N) \quad x_{hi,\alpha}^* = \frac{H_h S_i}{T} + \alpha \frac{S_i^2}{2I_h}(z_{hi} - \beta_{h,\alpha}^*(\boldsymbol{z}) - \rho_{i,\alpha}^*(\boldsymbol{z})), \tag{4.7}$$

where $(\beta_{h,\alpha}^*(\boldsymbol{z}))_{h\in C}$ and $(\rho_{i,\alpha}^*(\boldsymbol{z}))_{i\in N}$ are the Lagrange multipliers of the constraints in Ξ (dual solution). Imposing these constraints on the primal solution $(x_{hi,\alpha}^*)_{h\in C, i\in N}$ we obtain

$$(\forall i \in N) \quad S_i = \sum_{h\in C} x_{hi,\alpha}^* = S_i + \alpha \frac{S_i^2}{2} \sum_{h\in C} I_h^{-1}(z_{hi} - \beta_{h,\alpha}^*(\boldsymbol{z}) - \rho_{i,\alpha}^*(\boldsymbol{z})) \tag{4.8}$$

$$(\forall h \in C) \quad H_h = \sum_{i\in N} x_{hi,\alpha}^* = H_h + \alpha \frac{1}{2I_h} \sum_{i\in N} S_i^2(z_{hi} - \beta_{h,\alpha}^*(\boldsymbol{z}) - \rho_{i,\alpha}^*(\boldsymbol{z})), \tag{4.9}$$

which yields that, under the additional condition $\rho_1 = 0$, the dual solution is the unique solution to the system

$$(\forall h \in C) \quad \beta_h = \sum_{i\in N}(z_{hi} - \rho_i)\eta_i \tag{4.10}$$

$$(\forall i \in N) \quad \rho_i = \sum_{h\in C}(z_{hi} - \beta_h)\iota_h, \tag{4.11}$$

where, for every $h \in C$ and $i \in N$, $\iota_h = I_h^{-1}/\sum_{g\in C} I_g^{-1}$ and $\eta_i = S_i^2/\sum_{j\in N} S_j^2$. Hence, the dual solution $(\boldsymbol{\beta}^*, \boldsymbol{\rho}^*)$ does not depend on α and, therefore, from (4.7), α should be small enough for satisfying $x_{hi,\alpha}^*(\boldsymbol{z}) > 0$.

Finally, (4.6) follows from a straightforward computation. □

4.2 *Numerical example*

We provide an example in a fictitious city with high income segregation, where we obtain the inclusion optimum, we compare it with the equilibrium and we compute subsidies in order to reach this optimum. We show that the solution to the social inclusion problem is an allocation with better levels of socioeconomic homogeneity. In this example, we take arbitrary values for the supply, demand (satisfying market clearing condition), and utility.

Consider a city with 10 zones ($|N| = 10$) and 5 types of households ($|C| = 5$). The convergence criteria of the algorithm for solving problem (4.5) (see Appendix) is $\|\boldsymbol{b}_n - \boldsymbol{b}_{n+1}\|/\|\boldsymbol{b}_n\| \leq 10^{-10}$ and $\|\boldsymbol{r}_n - \boldsymbol{r}_{n+1}\|/\|\boldsymbol{r}_n\| \leq 10^{-10}$. The real estate supply per zone and the number of households per type are $S = (S_1, \ldots, S_{10}) = (25, 37, 24, 21, 34, 43, 23, 27, 20, 14)$ and $H = (H_1, \ldots, H_5) = (50, 56, 51, 60, 51)$, respectively, and the total supply (or demand) is $T = 268$. The average income of households per type is $I = (I_1, \ldots, I_5) = (2, 4, 6, 8, 10)$, which is used in this case as the segregation index in (4.2) and (4.3). Additionally, utilities $\boldsymbol{z} = (z_{hi})_{h\in C, i\in N}$, assumed exogenous in

h \ i	1	2	3	4	5	6	7	8	9	10
1	50	50	50	0	0	0	0	-50	-50	-50
2	50	50	0	0	0	0	0	-50	-50	-50
3	-50	-50	0	0	50	50	50	50	0	0
4	0	0	0	50	50	50	0	0	0	0
5	-50	-50	-50	0	0	0	50	50	50	50

Table 1.: Exogenous values for $(z_{hi})_{h\in C,\, i\in N}$.

this example, are presented in Table 1. We recall that, for every $h \in C$ and $i \in N$, z_{hi} represents the utility perceived by a household type h for a location in i. Then, for instance, households type $h = 1$ have preference for zones $1, 2$ and 3, and dislike $8, 9$ and 10.

Table 2 presents the unsubsidized equilibrium $\boldsymbol{x}(\boldsymbol{z}) = (x_{hi}(\boldsymbol{z}))_{h\in C,\, i\in N}$ obtained by solving (2.7) via the algorithm in Macgill (1977) with $\mu = 5 \times 10^{-2}$. It shows the income segregation level of the equilibrium by zone $SL_i(\boldsymbol{x}(\boldsymbol{z}))$ and the aggregated segregation level $SL(\boldsymbol{x}(\boldsymbol{z}))$, computed by (4.3). Additionally, in Figure 1 we show the proportion of households of each type $h \in \{1, \ldots, 5\}$ located in every zone $i \in \{1, \ldots, 10\}$ at the equilibrium. We observe very high segregation in all zones concentrating poor, middle-class and rich households in different zones, following the spatial distribution utilities in Table 1.

h \ i	1	2	3	4	5	6	7	8	9	10	$SL(\boldsymbol{x}(\boldsymbol{z}))$
1	9	13	19	2	2	3	2	0	0	0	
2	16	23	3	3	3	4	3	0	0	0	
3	0	0	1	1	11	14	9	13	1	1	
4	0	1	1	15	16	21	1	2	2	1	
5	0	0	0	1	1	1	9	12	16	11	
$SL_i(\boldsymbol{x}(\boldsymbol{z}))$	1.64	1.64	1.57	2.17	1.00	1.00	0.89	1.61	4.13	4.13	**19.78**

Table 2.: Equilibrium $\boldsymbol{x}(\boldsymbol{z})$ and segregation level.

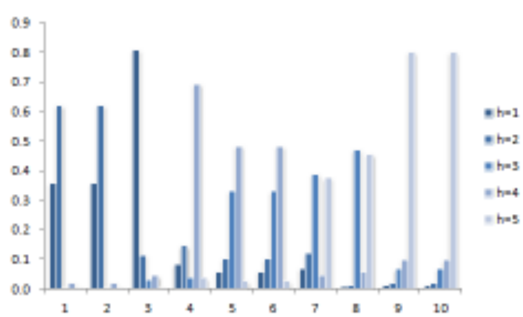

Figure 1.: Proportion of types of households by zone for the equilibrium.

In order to obtain a less segregated city, we consider the computation of the aggregated segregation level in terms of α, obtained from (4.6) in Proposition 4.2 for small values of

α, i.e., the planner gives high weight to inclusion in (4.5). It follows from Proposition 4.2, that parameters $(\boldsymbol{\beta}^*, \boldsymbol{\rho}^*)$ are independent of α and the aggregated segregation level is a quadratic and increasing function of α, which is verified numerically in Figure 2. In this

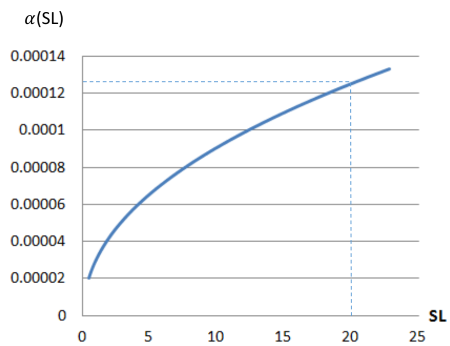

Figure 2.: Function $\alpha(SL)$.

figure, observe that an aggregated segregation level of 20 units of income (approximately that of the equilibrium $\boldsymbol{x}(\boldsymbol{z})$ in Table !!!) is obtained by using a value of $\alpha \approx 1.3 \times 10^{-4}$. By considering the much lower value of $\alpha = 3 \times 10^{-5}$, we obtain the solution $\boldsymbol{x}^*(\boldsymbol{z}) = (x^*_{hi}(\boldsymbol{z}))_{h \in C,\, i \in N}$ to the inclusion problem (4.5) from the algorithm detailed in the Appendix. The results and the corresponding segregation level in every zone are presented in Table 3. In Figure 3, we exhibit the proportion of households of each type $h \in \{1, \dots, 5\}$ located in every zone $i \in \{1, \dots, 10\}$. We observe that the initial segregation level of the equilibrium $\boldsymbol{x}(\boldsymbol{z})$ is drastically reduced in every zone: SL_i is in the interval $[0.89, 4.13]$ for the unsubsidized equilibrium, while for the inclusion social optimum the index vary in the interval $[0.01, 0.29]$ and SL reduces from 19.78 to 0.99. The reader can verify from Figure 2 that $\alpha(0.99) \approx 3 \times 10^{-5}$, which coincides with the theoretical square root relation for $\alpha(SL)$ provided in Proposition 4.2.

h \ i	1	2	3	4	5	6	7	8	9	10	$SL(\boldsymbol{x}^*)$
1	7	11	6	4	5	6	4	3	3	2	
2	7	13	5	4	6	7	4	4	3	2	
3	3	2	4	4	8	11	5	7	4	3	
4	5	7	5	5	9	12	4	5	4	3	
5	3	4	3	4	6	7	6	8	6	4	
$SL_i(\boldsymbol{x}^*)$	0.13	0.29	0.05	0.01	0.04	0.06	0.07	0.20	0.10	0.05	**0.99**

Table 3.: Solution $\boldsymbol{x}^*$ and segregation level.

We then compute subsidies for reaching the inclusion optimum $\boldsymbol{x}^*(\boldsymbol{z})$. In Table 4 and 5 we present subsidies, rents, and households' utilities considering two policies mentioned in Section 3.2. In Table 4 we show the resulting taxes and subsidies when the policy instrument is that *agent 1 is not affected*: no subsidies or taxes are implemented for this

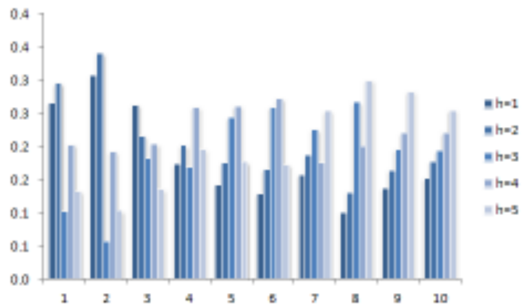

Figure 3.: Proportion of types of households by zone for the inclusion optimum.

type of household and the utility is not modified, i.e., $\beta_1 = b_1(\boldsymbol{z}) = 0$. Since $\beta_2, \beta_3, \beta_4$ and β_5 can be arbitrarily chosen by the planner, we set these values equal to 1. In Table 5 we present subsidies and taxes for the *self-funded policy by zone*. In this policy, the planner achieves the inclusion optimum without investing in subsidies or collecting taxes in any zone, she only make transfers between agents in each zone. For this case we choose $\eta = 0$ and $\boldsymbol{\beta} = (0, 1, 1, 1, -3)$ satisfying $\frac{1}{|C|}\sum_{h \in C} \beta_h = \eta$. Note that negative rents $(\rho_i)_{i \in N}$ can be made positive by choosing $\eta \leq -24$, however this reduces the average utilities of households.

h \ i	1	2	3	4	5	6	7	8	9	10	β_h
1	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	0
2	2	2	51	2	2	2	2	2	2	2	1
3	98	97	50	1	-47	-47	-48	-96	-48	-48	1
4	51	51	51	-46	-46	-45	3	-46	-46	-47	1
5	99	98	99	1	2	2	-47	-95	-96	-97	1

Table 4.: Subsidies for policy "Agent type 1 is not affected".

h \ i	1	2	3	4	5	6	7	8	9	10	β_h
1	-49	-49	-49	9	19	18	19	48	38	39	0
2	-47	-47	2	11	21	21	21	50	41	41	1
3	49	48	0	10	-29	-28	-29	-48	-9	-9	1
4	2	2	2	-37	-27	-27	21	2	-8	-8	1
5	46	45	46	7	16	16	-32	-52	-62	-62	-3
ρ_i	-16	-20	-16	-4	1	-1	5	-17	-24	-20	
$\sum_{h \in C} s_{hi}^2$	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	

Table 5.: Subsidies for policy "Self funded policy by zone".

5. Conclusions

To the best of our knowledge, until now the land use planning and subsidy problems defined in a discrete domain of zones and households was open, so planners neither had a method to identify the optimum allocation of households for specific objectives, nor a procedure to calculate optimal subsidies.

The main contribution of this paper is to tackle the difficulty of the interdependence among subsidies, rents and households' utilities, and location externalities associated to a specific social goal. Our approach simplify this problem by showing that it can be split into the planning problem and the optimal subsidy problem.

We prove that the planning problem is independent of subsidies. Additionally, the optimal subsidy problem can be also divided, identifying a feasible set of subsidies reaching the social optimal allocation and identifying an optimal political instrument.

We obtain the fundamental result that the optimal allocation of any social objective function can be decentralized by a market equilibrium with feasible subsidies (Theorem 3.3). Additionally, our approach gives the planner the flexibility of choosing alternative political instruments to define optimal subsidies; some examples are provided in Section 3.2.

As an interesting example, we analyze the social inclusion problem and we simulate a prototype city showing how our methodology works. For this example we discuss policy instruments that represent two real situations faced by the planner: difficulty on subsidizing/taxing some type of household and a budget constraint.

The two main limitations of our approach is the assumption that transport costs, supply and demand in the land use market are exogenous. Integrating the transportation subsystem in the land use planning problem leads to a fixed point: the social optimal allocation influences the transportation system and, in turn, the new transportation costs modifies the optimal allocation. Including endogenous supply and/or demand in the land use market can be tackled by formulating a long-term land use planning problem. This phenomena appear in LUT equilibrium models and deserves further research to extend the equilibrium properties found in Bravo et al. (2010) to other social goals. In the absence of such properties, bi-level heuristics may be applied. It is important to note that both extensions are confined to the formulation of the planning problem, while our splitting approach is valid once the optimal planning on the integrated LUT problem is solved.

References

Águila, L.F., 2006. *Modelo Operativo de Planificación Óptima de Subsidios en Sistemas Urbanos*, MSc. Thesis, Universidad de Chile.

Anas, A., 1982. *Residential location markets and urban transportation*. Academic Press, London, England.

Anas, A., 1983. Discrete choice theory, information theory and the multinomial logit and gravity models. *Transportation Research Part B: Methodological*, 17(1):13–23.

Bauschke, H.H., and Combettes, P.L. 2011. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* New York: Springer-Verlag.

Bravo, M., Briceño-Arias, L.M., Cominetti, R., Cortés, C.E., and Martínez, F.J. 2010. "An integrated behavioral model of the land-use and transport systems with network congestion and location externalities." *Transportation Research Part B: Methodological* 44(4): 584–596.

Briceño-Arias, L.M., and Combettes, P.L. 2011. "A monotone+skew splitting model for composite monotone inclusions in duality." *SIAM Journal on Optimization* 21: 1230–1250.

Briceño-Arias, L.M., Cominetti, R., Cortés, C.E., and Martínez, F.J. 2008. "An integrated behavioral model of land use and transport system: a hyper-network equilibrium approach." *Networks and Spatial Economics* 8: 201–224.

Combettes, P.L., and Wajs, V.R. 2005. “Signal recovery by proximal forward-backward splitting.” *SIAM Journal on Multiscale Modeling and Simulation* 4: 1168–1200.
Ellickson, B. 1981. An alternative test of the hedonic theory of housing markets. *Journal on Urban Economics* 9(1):56–79.
Hiriart-Urruty, J.-B., and Lemaréchal, C. 1993. *Convex analysis and minimization algorithms I.* Berlin Heidelberg: Springer-Verlag.
Hunt, J.D., Kriger, D.S., and Miller, E.J. 2005. “Current operational urban land-use-transport modelling frameworks: a review.” *Transport Reviews* 25(3): 329–376.
Ma, X. and Lo, H.K., 2012. Modeling transport management and land use over time. *Transportation Research Part B: Methodological*, 46(6): 687–709.
Macgill, S.M. 1977. ”Theoretical properties of biproportional matrix adjustments.”*Environment and Planning A*. 9(6): 687-701.
Martínez, F. 1992. The Bid-Choice land use model: an integrated economic framework. *Environmental and Planning A* 24(6):871–885.
Martínez, F., Henríquez, R., 2007. A random bidding and supply land use model. *Transportation Research Part B: Methodological.* 41(6): 632–651.
Ng, Lo...
Preston, J., Simmonds, D., and Pagliara, F. 2010. *Residential Location Choice: Models and Applications.* Berlin Heidelberg: Springer-Verlag.
Rockafellar, R.T. 1970. *Convex analysis.* Princeton Mathematical Series, 28. Princeton: Princeton University Press.
Timmermans, H.J.P., and Zhang, J. 2009. “Modeling household activity travel behavior: Example of the state of the art modeling approaches and research agenda.” *Transportation Research Part B: Methodological* 43: 187–190.
Wegener, M. 1994. “Operational urban models: State of the art.” *Journal of the American Planning Association* 60(1): 17–29.
Wegener, M. 1998. “Models of urban land use, transport and environment.” In *Network Infrastructure and the Urban Environment. Advances in Spatial Science*, edited by L. Lundqvist, L.-G. Mattsson, and T.J. Kim. Berlin Heidelberg: Springer-Verlag.
Ying, J. Q. 2015. “Optimization for Multiclass Residential Location Models with Congestible Transportation Networks.” *Transportation Science* 49(3):452–471.

6. Appendix

6.1 *Notation and preliminaries*

Let $(\mathcal{H}, \|\cdot\|)$ be a finite dimensional Euclidean space and denote by $\Gamma_0(\mathcal{H})$ the family of lower semicontinuous convex functions $\varphi\colon \mathcal{H} \to \,]-\infty, +\infty]$ such that $\operatorname{dom}\varphi = \big\{x \in \mathcal{H} \mid \varphi(x) < +\infty\big\} \neq \varnothing$. A function $\varphi\colon \mathcal{H} \to \,]-\infty, +\infty]$ is coercive if $\lim_{\|x\|\to+\infty} \varphi(x) = +\infty$. Now let $\varphi \in \Gamma_0(\mathcal{H})$. The conjugate of φ is the function $\varphi^* \in \Gamma_0(\mathcal{H})$ defined by $\varphi^*\colon u \mapsto \sup_{x\in\mathcal{H}}(\langle x \mid u\rangle - \varphi(x))$. Moreover, for every $x \in \mathcal{H}$, $\varphi + \|x - \cdot\|^2/2$ possesses a unique minimizer, which is denoted by $\operatorname{prox}_\varphi x$. Alternatively,

$$\operatorname{prox}_\varphi = (\mathrm{Id} + \partial\varphi)^{-1}, \tag{6.1}$$

where

$$\partial\varphi\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H}) \quad \langle y - x \mid u\rangle + \varphi(x) \le \varphi(y)\big\} \tag{6.2}$$

is the subdifferential of φ. In the particular case when φ is differentiable in some subset C of $\mathcal{H}$, we have, for every $x \in C$, $\partial\varphi(x) = \{\nabla\varphi(x)\}$. For every convex subset C of $\mathcal{H}$, the indicator function of C, denoted by ι_C, is the function which is 0 in C and $+\infty$ in $\mathcal{H} \setminus C$.

The following result will be useful in the following sections and some parts of it can be derived from the proof given in Rockafellar (1970, Corollary 26.3.1).

Lemma 6.1. *Let $\psi\colon \operatorname{dom}\psi \subset \mathrm{R} \to \left]-\infty,+\infty\right]$ be strictly convex, differentiable in $\operatorname{int}\operatorname{dom}\psi$, and such that $\operatorname{ran}(\psi') = \mathrm{R}$. Then, ψ^* is strictly convex, differentiable in $\operatorname{dom}\psi^* = \mathrm{R}$, and $\operatorname{ran}\psi^* \subset \operatorname{dom}\psi$. Moreover,*

$$\psi^*\colon \eta \mapsto (\psi')^{-1}(\eta)\eta - \psi\big((\psi')^{-1}(\eta)\big) \quad \textit{and} \quad (\psi^*)' = (\psi')^{-1}. \tag{6.3}$$

6.2 *Dual problem computation*

For computing the dual formulation of Problem 2.1, we need the following definitions and preliminaries. Define

$$\begin{cases} \boldsymbol{\Psi}\colon \mathrm{R}^{|C|\times|N|} \to \left]-\infty,+\infty\right] \\ \qquad \boldsymbol{x} \quad \mapsto \quad \begin{cases} \displaystyle\sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},x_{hi}), & \text{if } \boldsymbol{x}\in \displaystyle\bigtimes_{h\in C}\bigtimes_{i\in N}\operatorname{dom}\psi_{hi}(z_{hi},\cdot) \\ +\infty, & \text{otherwise} \end{cases} \\ \boldsymbol{\Lambda}\colon \mathrm{R}^{|C|\times|N|} \to \mathrm{R}^{|C|+|N|} \\ \qquad \boldsymbol{x} \quad \mapsto \quad \left(\left(\displaystyle\sum_{i\in N}x_{hi}\right)_{h\in C}, \left(\displaystyle\sum_{h\in C}x_{hi}\right)_{i\in N}\right), \end{cases} \tag{6.4}$$

where $\boldsymbol{x} = (x_{hi})_{h\in C,\, i\in N}$ is a generic element of $\mathrm{R}^{|C|\times|N|}$.

Proposition 6.2 (Properties of $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$)**.** *Let $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ be defined as in* (6.4)*. Then, the following statements hold.*

(i) *$\boldsymbol{\Psi}$ is strictly convex, coercive, and*

$$(\forall\gamma\in\left]0,+\infty\right[) \quad \operatorname{prox}_{\gamma\boldsymbol{\Psi}} = \big(\operatorname{prox}_{\gamma\psi_{hi}(z_{hi},\cdot)}\big)_{h\in C,\, i\in N}. \tag{6.5}$$

(ii) *We have*

$$\boldsymbol{\Psi}^*\colon \mathrm{R}^{|C|\times|N|} \to \left]-\infty,+\infty\right] : \boldsymbol{u} \mapsto \sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},u_{hi}), \tag{6.6}$$

where, for every $h\in C$ and $i\in N$,

$$\varphi_{hi}\colon (z_{hi},u) \mapsto \psi_{hi}(z_{hi},\cdot)^*(u) = \sup_{x\in[0,a_{hi}[}\left(\langle x \mid u\rangle - \psi_{hi}(z_{hi},x)\right) \tag{6.7}$$

is differentiable. Moreover, $\boldsymbol{\Psi}^*$ is differentiable and $\nabla\boldsymbol{\Psi}^* = (\varphi_{hi}(z_{hi},\cdot)')_{h\in C, i\in N}$. In addition, suppose that, for every $h\in C$ and $i\in N$, $\psi_{hi}(z_{hi},\cdot)$ is differentiable in $\left]0,a_{hi}\right[$ and $\operatorname{ran}(\psi_{hi}(z_{hi},\cdot)') = \mathrm{R}$. Then, for every $h\in C$ and $i\in N$,

$$(\forall u\in\mathrm{R}) \quad \varphi_{hi}(z_{hi},u) = (\psi_{hi}(z_{hi},\cdot)')^{-1}(u)u - \psi_{hi}\big(z_{hi},(\psi_{hi}(z_{hi},\cdot)')^{-1}(u)\big), \tag{6.8}$$

$\varphi_{hi}(z_{hi},\cdot)' = (\psi_{hi}(z_{hi},\cdot)')^{-1}$, and $\boldsymbol{\Psi}^*$ is strictly convex.

(iii) *$\boldsymbol{\Lambda}$ is linear, bounded, $\boldsymbol{\Lambda}^*\colon (\boldsymbol{b},\boldsymbol{r}) \mapsto (b_h + r_i)_{h\in C, i\in N}$, where $(\boldsymbol{b},\boldsymbol{r}) = ((b_h)_{h\in C},(r_i)_{i\in N})$ is a generic element of $\mathrm{R}^{|C|+|N|}$, and $\|\boldsymbol{\Lambda}\| = \sqrt{|C|+|N|}$.*

Proof. (i): Is a consequence of (6.4), the properties of the class $\mathscr{C}$ in (2.1), and Combettes and Wajs (2005, Lemma 2.9). (ii): It follows from Bauschke and Combettes (2011, Proposition 13.27) that $\boldsymbol{\Psi}^* = \sum_{h\in C}\sum_{i\in N}\psi_{hi}(z_{hi},\cdot)^* = \sum_{h\in C}\sum_{i\in N}\varphi_{hi}(z_{hi},\cdot)$. The differentiability follows from Hiriart-Urruty and Lemaréchal (1993, Proposition 6.2.1) and the last result follows from Lemma 6.1. (iii): It is clear that $\boldsymbol{\Lambda}$ is linear and bounded. For every $\boldsymbol{x} \in \mathrm{R}^{|C|\times|N|}$ and $(\boldsymbol{b}, \boldsymbol{r}) \in \mathrm{R}^{|C|+|N|}$, we have

$$\begin{aligned} \langle (\boldsymbol{b}, \boldsymbol{r}) \mid \boldsymbol{\Lambda x} \rangle &= \sum_{h\in C} b_h \Big(\sum_{i\in N} x_{hi} \Big) + \sum_{i\in N} r_i \Big(\sum_{i \in N} x_{hi} \Big) \\ &= \sum_{h\in C} \sum_{i\in N} x_{hi}(b_h + r_i) \\ &= \langle \boldsymbol{\Lambda}^*(\boldsymbol{b}, \boldsymbol{r}) \mid \boldsymbol{x} \rangle. \end{aligned} \tag{6.9}$$

On the other hand, using the inequality $2xy \leq x^2 + y^2$ we obtain, for every $\boldsymbol{x} \in \mathrm{R}^{|C|\times|N|}$,

$$\begin{aligned} \|\boldsymbol{\Lambda x}\|^2 &= \sum_{h\in C} \Big(\sum_{i\in N} x_{hi} \Big)^2 + \sum_{i\in N} \Big(\sum_{h\in C} x_{hi} \Big)^2 \\ &= \sum_{h\in C} \Big(\sum_{i\in N} x_{hi}^2 + \sum_{j\neq i} 2x_{hi}x_{hj} \Big) + \sum_{i\in N} \Big(\sum_{h\in C} x_{hi}^2 + \sum_{g\neq h} 2x_{hi}x_{gi} \Big) \\ &\leq \sum_{h\in C} \Big(\sum_{i\in N} x_{hi}^2 + (|N|-1) \sum_{i\in N} x_{hi}^2 \Big) + \sum_{i\in N} \Big(\sum_{h\in C} x_{hi}^2 + (|C|-1) \sum_{h\in C} x_{hi}^2 \Big) \\ &= (|C| + |N|)\|\boldsymbol{x}\|^2, \end{aligned} \tag{6.10}$$

which yields $\|\boldsymbol{\Lambda}\| \leq \sqrt{|C| + |N|}$. The equality follows by taking, in particular, for every $(h, i) \in C \times N$, $x_{hi} = 1$, which yields

$$\|\boldsymbol{\Lambda x}\|^2 = \sum_{h\in C} |N|^2 + \sum_{i\in N} |C|^2 = (|C| + |N|)|C||N| = (|C| + |N|)\|\boldsymbol{x}\|^2. \tag{6.11}$$

Hence $\|\boldsymbol{\Lambda}\| = \sqrt{|C| + |N|}$. □

Proof of Proposition 2.2 Indeed, Problem 2.1 can be written equivalently as

$$\underset{\substack{\boldsymbol{x}\in \mathrm{R}^{|C|\times|N|} \\ \boldsymbol{\Lambda x} = (\boldsymbol{H}, \boldsymbol{S})}}{\text{minimize}} \; \boldsymbol{\Psi}(\boldsymbol{x}), \tag{6.12}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ are defined in (2.4). Therefore, from Bauschke and Combettes (2011, Proposition 19.19) we have that the dual problem is

$$\underset{(\boldsymbol{b}, \boldsymbol{r})\in \mathrm{R}^{|C|+|N|}}{\text{minimize}} \; \boldsymbol{\Psi}^*(-\boldsymbol{\Lambda}^*(\boldsymbol{b}, \boldsymbol{r})) + \langle (\boldsymbol{b}, \boldsymbol{r}) \mid (\boldsymbol{H}, \boldsymbol{S}) \rangle, \tag{6.13}$$

or equivalently, from Proposition 6.2(ii)–(iii),

$$\underset{(\boldsymbol{b}, \boldsymbol{r})\in \mathrm{R}^{|C|+|N|}}{\text{minimize}} \; \sum_{h\in C} \sum_{i\in N} \varphi_{hi}(z_{hi}, -b_h - r_i) + \sum_{h\in C} H_h b_h + \sum_{i\in N} S_i r_i, \tag{6.14}$$

and the proof is finished.

Proof of Proposition 2.3 Since $\boldsymbol{\Psi} \in \Gamma_0(\mathrm{R}^{|C|\times|N|})$ is coercive, Ξ is closed and convex, and (2.3) yields $\Xi \cap \operatorname{dom} \boldsymbol{\Psi} \neq \varnothing$, Bauschke and Combettes (2011, Proposition 11.4(i)) asserts that the primal problem has solutions. It follows from (6.12) that Problem 2.1

can be written equivalently as

$$\underset{\boldsymbol{x}\in\mathrm{R}^{|C|\times|N|}}{\text{minimize}}\;\; \boldsymbol{\Psi}(\boldsymbol{x})+\iota_{\{(\boldsymbol{H},\boldsymbol{S})\}}(\boldsymbol{\Lambda x}). \tag{6.15}$$

Note that $\iota_{\{(\boldsymbol{H},\boldsymbol{S})\}}\in\Gamma_0(\mathrm{R}^{|C|+|N|})$ is polyhedral. Now, since $(\boldsymbol{H},\boldsymbol{S})\in\,]0,+\infty[^{|C|+|N|}$, it follows from (2.3) and (6.4) that

$$(\boldsymbol{H},\boldsymbol{S})\in\operatorname{int}\left(\boldsymbol{\Lambda}\left(\bigtimes_{h\in C}\bigtimes_{i\in N}[0,a_{hi}[\right)\right)=\bigtimes_{h\in C}\,]0,a_h[\,\times\bigtimes_{i\in N}\,]0,a_i[\,, \tag{6.16}$$

where, for every $h\in C$, $a_h=\sum_{i\in I}a_{hi}$ and, for every $i\in N$, $a_i=\sum_{h\in C}a_{hi}$. Hence, from Bauschke and Combettes (2011, Fact 15.25) we have $\inf(\boldsymbol{\Psi}+\iota_{\{(\boldsymbol{H},\boldsymbol{S})\}}\circ\boldsymbol{\Lambda})=-\min(\boldsymbol{\Psi}^*\circ-\boldsymbol{\Lambda}^*+\iota^*_{\{(\boldsymbol{H},\boldsymbol{S})\}})$. Therefore, we have existence of solutions to the dual problem. Moreover, it follows from Proposition 6.2(ii) that $\boldsymbol{\Psi}^*$ is differentiable in $\mathrm{R}^{|C|+|N|}$. Altogether, Bauschke and Combettes (2011, Proposition 19.3) asserts that Problem 2.1 has a unique solution

$$\overline{\boldsymbol{x}}=\nabla\boldsymbol{\Psi}^*(\boldsymbol{\Lambda}^*(\overline{\boldsymbol{b}},\overline{\boldsymbol{r}})), \tag{6.17}$$

where $(\overline{\boldsymbol{b}},\overline{\boldsymbol{r}})$ is a solution to the dual problem (2.5). Moreover, it follows from Lemma 6.1 and Proposition 6.2(iii) that (6.17) is equivalent to

$$(\forall h\in C)(\forall i\in N)\quad \overline{x}_{hi}=(\psi_{hi}(z_{hi},\cdot)^*)'(\overline{b}_h+\overline{r}_i)=\big(\varphi_{hi}(z_{hi},\cdot)\big)'(\overline{b}_h+\overline{r}_i). \tag{6.18}$$

Finally let us prove that, under one of the constraints (i)–(iv), $\boldsymbol{\Phi}$ is strictly convex and, hence, the dual problem (2.5) has a unique solution. Indeed, it follows from Lemma 6.1 that, for every $h\in C$ and $i\in N$, φ_{hi} is strictly convex. Let $(\boldsymbol{b}^1,\boldsymbol{r}^1)\neq(\boldsymbol{b}^2,\boldsymbol{r}^2)$ be vectors in $\mathrm{R}^{|C|+|N|}$ and let $\alpha\in\,]0,1[$. We have

$$\begin{aligned}
\boldsymbol{\Phi}(\alpha(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)(\boldsymbol{b}^2,\boldsymbol{r}^2)) &= \sum_{h\in C}H_h(\alpha b_h^1+(1-\alpha)b_h^2)+\sum_{i\in N}S_i(\alpha r_i^1+(1-\alpha)r_i^2)\\
&\quad+\sum_{h\in C}\sum_{i\in N}\varphi_{hi}\big(z_{hi},-(\alpha b_h^1+(1-\alpha)b_h^2)-(\alpha r_i^1+(1-\alpha)r_i^2)\big)\\
&=\alpha\Bigg(\sum_{h\in C}H_hb_h^1+\sum_{i\in N}S_ir_i^1\Bigg)+(1-\alpha)\Bigg(\sum_{h\in C}H_hb_h^2+\sum_{i\in N}S_ir_i^2\Bigg)\\
&\quad+\sum_{h\in C}\sum_{i\in N}\varphi_{hi}\big(z_{hi},\alpha(-b_h^1-r_i^1)+(1-\alpha)(-b_h^2-r_i^2)\big).
\end{aligned} \tag{6.19}$$

Since, for every $h\in C$ and $i\in N$, $\varphi_{hi}(z_{hi},\cdot)$ is strictly convex, it is enough to prove that, under one of the constraints (i)–(iv), there exist $h_0\in C$ and $i_0\in N$ such that $-b_{h_0}^1-r_{i_0}^1\neq-b_{h_0}^2-r_{i_0}^2$, in which case from (6.19) we obtain that

$$\boldsymbol{\Phi}(\alpha(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)(\boldsymbol{b}^2,\boldsymbol{r}^2))<\alpha\boldsymbol{\Phi}(\boldsymbol{b}^1,\boldsymbol{r}^1)+(1-\alpha)\boldsymbol{\Phi}(\boldsymbol{b}^2,\boldsymbol{r}^2), \tag{6.20}$$

and the result follows. Let us proceed by contradiction. Suppose that

$$(\forall h\in C)(\forall i\in N)\quad -b_h^1-r_i^1=-b_h^2-r_i^2. \tag{6.21}$$

If (i) holds, we have $b_1^1 = b_1^2 = \eta$ and we deduce from (6.21) in the particular case $h = 1$ that, for every $i \in N$, $r_i^1 = r_i^2$. Hence, it follows again from (6.21) that, for every $h \in C \setminus \{1\}$, $b_h^1 = b_h^2$, which contradicts $(\boldsymbol{b}^1, \boldsymbol{r}^1) \neq (\boldsymbol{b}^2, \boldsymbol{r}^2)$. Now suppose that (ii) holds. Then we have $\sum_{h\in C} b_h^1 = \sum_{h\in C} b_h^2 = \eta$ and, by summing in h in (6.21), we deduce, for every $i \in N$, $r_i^1 = r_i^2$. The contradiction is obtained in the same way as before. The cases (iii) and (iv) are analogous.

6.3 *Planning algorithms*

Proposition 6.3 (Planning problem algorithm)**.** *For every $h \in C$ and $i \in N$, let $(e_{hi,n})_{n\in\mathrm{N}}$ be an absolutely summable sequence in R, let $x_{hi,0} \in \mathrm{R}$, let $(b_{h,0}, r_{i,0}) \in \mathrm{R}^2$, let $\varepsilon \in \left]0, 1/(\sqrt{|C| + |N|} + 1)\right[$, let $(\gamma_n)_{n\in\mathrm{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\sqrt{|C|+|N|}]$, and set*

$$(\forall n \in \mathrm{N}) \quad \left\lfloor \begin{array}{l} \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} y_{1hi,n} = x_{hi,n} - \gamma_n(b_{h,n} + r_{i,n}) \\ p_{1hi,n} = \operatorname{prox}_{\gamma_n \psi_{hi}(z_{hi},\cdot)} y_{1hi,n} + e_{hi,n} \end{array} \right. \\ \text{For every } h \in C \\ \left\lfloor \begin{array}{l} p_{2h,n} = b_{h,n} + \gamma_n\big(\sum_{i\in N} x_{hi,n} - H_h\big) \\ b_{h,n+1} = b_{h,n} + \gamma_n\big(\sum_{i\in N} p_{1hi,n} - H_h\big) \end{array} \right. \\ \text{For every } i \in N \\ \left\lfloor \begin{array}{l} p_{2i,n} = r_{i,n} + \gamma_n\big(\sum_{h\in C} x_{hi,n} - S_i\big) \\ r_{i,n+1} = r_{i,n} + \gamma_n\big(\sum_{h\in C} p_{1hi,n} - S_i\big) \end{array} \right. \\ \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} q_{hi,n} = p_{1hi,n} - \gamma_n(p_{2h,n} + p_{2i,n}) \\ x_{hi,n+1} = x_{hi,n} - y_{1hi,n} + q_{hi,n} \end{array} \right. \end{array} \right. \tag{6.22}$$

Then the following statements hold for the solution $((\overline{x}_{hi})_{h\in C})_{i\in N}$ to Problem 2.1 and some solution $\big((\overline{b}_h)_{h\in C}, (\overline{r}_i)_{i\in N}\big)$ to its dual in (2.5).

(i) *For every $h \in C$ and $i \in N$, $x_{hi,n} - p_{1hi,n} \to 0$, $b_{h,n} - p_{2h,n} \to 0$, and $r_{i,n} - p_{2i,n} \to 0$.*

(ii) *For every $h \in C$ and $i \in N$, $x_{hi,n} \to \overline{x}_{hi}$, $p_{1hi,n} \to \overline{x}_{hi}$, $b_{h,n} \to \overline{b}_h$, $p_{2h,n} \to \overline{b}_h$, $r_{i,n} \to \overline{r}_i$, and $p_{2i,n} \to \overline{r}_i$.*

Proof. See Briceño-Arias and Combettes (2011). □

The difficulty of the algorithm proposed in Proposition 6.3 lies in the computation, for every $h \in C$, $i \in N$, and $n \in \mathrm{N}$, of $\operatorname{prox}_{\gamma_n \psi_{hi}(z_{hi},\cdot)}$. Several examples in which the proximity operator can be computed explicitly can be found in Combettes and Wajs (2005). The following result shows some interesting cases in which an explicit computation of the proximity operator can be obtained.

Lemma 6.4. *Let $z \in \mathrm{R}$, $a \in]0, +\infty[$, $b \in]0, +\infty[$, $\gamma \in]0, +\infty[$, and $\mu \in]0, +\infty[$.*

(i) *Let $\psi\colon (z, x) \mapsto -zx + \frac{1}{\mu}x(\ln x - 1)$. Then $\psi \in \mathscr{C}$ and $\operatorname{prox}_{\gamma\psi(z,\cdot)}\colon x \mapsto \frac{\gamma}{\mu}W(\frac{\mu}{\gamma}e^{\mu(x/\gamma+z)})$, where W is the product log function.*

(ii) *Let $\psi\colon (z, x) \mapsto \iota_{[0,+\infty[} - zx + a(x-b)^2$. Then $\psi \in \mathscr{C}$ and $\operatorname{prox}_{\gamma\psi(z,\cdot)}\colon x \mapsto \max\{(x + \gamma z + 2\gamma ab)/(1 + 2\gamma a), 0\}$.*

Proof. Let $(x, p) \in \mathrm{R}^2$. It is clear from (2.1) that both functions are in $\mathscr{C}$. (i): We have $p = \operatorname{prox}_{\gamma\psi(z,\cdot)} x \;\Leftrightarrow\; x - p = \gamma\psi(z,\cdot)'(p) \;\Leftrightarrow\; x + \gamma z = p + \frac{\gamma}{\mu}\ln p \;\Leftrightarrow\; \frac{\mu}{\gamma}x + \mu z = \frac{\mu}{\gamma}p + \ln p \;\Leftrightarrow\; e^{\mu(x/\gamma+z)} = pe^{\frac{\mu}{\gamma}p} \;\Leftrightarrow\; \frac{\mu}{\gamma}p = W(\frac{\mu}{\gamma}e^{\mu(x/\gamma+z)})$, and the result follows. (ii): We have $p = \operatorname{prox}_{\gamma\psi(z,\cdot)} x \;\Leftrightarrow\; x - p \in \gamma\partial\psi(z,\cdot)(p) \;\Leftrightarrow\; x - p \in N_{[0,+\infty[}(p) - \gamma z + 2\gamma a(p-b) \;\Leftrightarrow$

$x+\gamma z+2\gamma ab \in N_{[0,+\infty[}(p)+p(1+2\gamma a) \Leftrightarrow (x+\gamma z+2\gamma ab)/(1+2\gamma a) \in N_{[0,+\infty[}(p)+p \Leftrightarrow p = P_{[0,+\infty[}((x+\gamma z+2\gamma ab)/(1+2\gamma a))$, which yields the result. □

Example 6.5. *(Social inclusion algorithm) As an example of Proposition 6.3, we consider the social inclusion problem* (4.5) *by using the algorithm proposed in* (6.22)*, which, by applying Lemma 6.4(ii), becomes (we set* $e_{hi,n} \equiv 0$*)*

$$(\forall n \in \mathrm{N}) \quad \left\lfloor \begin{array}{l} \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} y_{1hi,n} = x_{hi,n} - \gamma_n(b_{h,n} + r_{i,n}) \\ p_{1hi,n} = \max\left\{ \dfrac{y_{1hi,n} + \gamma_n z_{hi} + \frac{2\gamma_n I_h H_h}{\alpha S_i T}}{1 + \frac{2\gamma_n I_h}{\alpha S_i^2}}, 0 \right\} \end{array} \right. \\ \text{For every } h \in C \\ \left\lfloor \begin{array}{l} p_{2h,n} = b_{h,n} + \gamma_n\big(\sum_{i \in N} x_{hi,n} - H_h\big) \\ b_{h,n+1} = b_{h,n} + \gamma_n\big(\sum_{i \in N} p_{1hi,n} - H_h\big) \end{array} \right. \\ \text{For every } i \in N \\ \left\lfloor \begin{array}{l} p_{2i,n} = r_{i,n} + \gamma_n\big(\sum_{h \in C} x_{hi,n} - S_i\big) \\ r_{i,n+1} = r_{i,n} + \gamma_n\big(\sum_{h \in C} p_{1hi,n} - S_i\big) \end{array} \right. \\ \text{For every } h \in C \text{ and } i \in N \\ \left\lfloor \begin{array}{l} q_{1hi,n} = p_{1hi,n} - \gamma_n(p_{2h,n} + p_{2i,n}) \\ x_{hi,n+1} = x_{hi,n} - y_{1hi,n} + q_{1hi,n}. \end{array} \right. \end{array} \right. \tag{6.23}$$

If the sequence $(\gamma_n)_{n\in\mathrm{N}}$ *is in* $\left]0, (|C|+|N|)^{-1/2}\right[$*, Proposition 6.3 asserts that, for every* $h \in C$ *and* $i \in N$*, the sequence* $(x_{hi,n})_{n\in\mathrm{N}}$ *converges to some* x^*_{hi} *and* $\boldsymbol{x}^* = (x^*_{hi})_{h\in C,\, i\in N}$ *is the solution to* (4.5) *and, additionally, the sequence* $(b_{h,n}, r_{i,n})_{n\in\mathrm{N}}$ *converges to some* (b^*_h, r^*_i) *and* $((b^*_h)_{h\in C}, (r^*_i)_{i\in N})$ *is a solution to the associated dual problem.*