OUTER APPROXIMATION METHOD FOR CONSTRAINED COMPOSITE FIXED POINT PROBLEMS INVOLVING LIPSCHITZ PSEUDO CONTRACTIVE OPERATORS *

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Abstract

We propose a method for solving constrained fixed point problems involving compositions of Lipschitz pseudo contractive and firmly nonexpansive operators in Hilbert spaces. Each iteration of the method uses separate evaluations of these operators and an outer approximation given by the projection onto a closed half-space containing the constraint set. Its convergence is established and applications to monotone inclusion splitting and constrained equilibrium problems are demonstrated.

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1 Introduction

The problem under consideration in this paper is the following.

Problem 1.1 Let $\mathcal{H}$ be a real Hilbert space, fix $\varepsilon \in]0,1[$, and let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $]0, 1-\varepsilon]$. For every $n \in \mathbb{N}$, let $T_n\colon \mathcal{H} \to \mathcal{H}$ be a firmly nonexpansive operator, let $R_n\colon \operatorname{dom} R_n \subset \mathcal{H} \to \mathcal{H}$ be a pseudo contraction such that $(\operatorname{Id} - R_n)$ is β_n–Lipschitzian, and let S be a closed convex subset of $\mathcal{H}$. The problem is to

$$\text{find} \quad x \in S \quad \text{such that} \quad (\forall n \in \mathbb{N}) \quad T_n R_n x = x. \tag{1.1}$$

The set of solutions to (1.1) is denoted by Z.

As will be seen subsequently, this formulation models a broad range of problems in numerical analysis, including monotone inclusions, variational inequalities, and equilibrium problems (see [1, 2] and the references therein). Methods can be found in the literature to solve Problem 1.1 in special cases. Thus, when $S = \mathcal{H}$, $R_n \equiv \operatorname{Id}$, and $Z \neq \varnothing$, algorithms can be found in [1, 3], and when $S = \mathcal{H}$, $T_n \equiv \operatorname{Id}$, and $R_n \equiv R$, where R is a Lipschitzian pseudo contraction from a convex set C into itself, methods can be found in [4, 5, 6, 7]. Since the composition between a firmly nonexpansive operator and a Lipschitzian pseudo contraction is not a pseudo contraction in general, Problem 1.1 can not be solved by the methods mentioned above. The purpose of the present paper is to provide an algorithm for solving Problem 1.1. It involves four elementary steps at each iteration n: the first three steps are successive computations of operators R_n, T_n, and R_n, and the last step is an outer approximation of the constraint. The latter is given by the projection onto a half-space containing S. In Section 2 we propose our algorithm and we prove its weak convergence to a solution to Problem 1.1. In Section 3 we study an application to monotone inclusions under convex constraints, and obtain an extension of a result of [8]. Finally, in Section 4, we study an application to equilibrium problems with convex constraints.

Notation 1.2 Throughout this paper $\mathcal{H}$ denotes a real Hilbert space, $\langle \cdot \mid \cdot \rangle$ denotes its scalar product, and $\|\cdot\|$ denotes the associated norm. For a single-valued operator $R\colon \operatorname{dom} R \subset \mathcal{H} \to \mathcal{H}$, the set of fixed points is $\operatorname{Fix} R = \big\{x \in \mathcal{H} \;\big|\; x = Rx\big\}$, R is χ–Lipschitzian for some $\chi \in]0, +\infty[$, if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\| \leq \chi \|x - y\|, \tag{1.2}$$

R is pseudo contractive if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\|^2 \leq \|x - y\|^2 + \|(\operatorname{Id} - R)x - (\operatorname{Id} - R)y\|^2, \tag{1.3}$$

R is firmly nonexpansive if it satisfies

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \|Rx - Ry\|^2 \leq \|x - y\|^2 - \|(\operatorname{Id} - R)x - (\operatorname{Id} - R)y\|^2, \tag{1.4}$$

or equivalently,

$$(\forall x \in \operatorname{dom} R)(\forall y \in \operatorname{dom} R) \quad \langle x - y \mid Rx - Ry \rangle \geq \|Rx - Ry\|^2, \tag{1.5}$$

and R is χ–cocoercive (or χ-inverse-strongly-monotone) if χR is firmly nonexpansive.

2 Algorithm and convergence

At each iteration $n \in \mathbb{N}$, our method for solving Problem 1.1 involves an outer approximation to S and separate computations of the operators T_n and R_n. Each approximation is computed by the projection onto a closed affine half-space containing S, and errors on the computation of the operators are modeled by the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$.

Algorithm 2.1 Let $(T_n)_{n\in\mathbb{N}}$, $(R_n)_{n\in\mathbb{N}}$, and S be as in Problem 1.1. For every $n \in \mathbb{N}$, let $Q_n : \mathcal{H} \to \mathcal{H}$ be the projector operator onto a closed affine half-space containing S, let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(c_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$, $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$, and $\sum_{n\in\mathbb{N}} \|c_n\| < +\infty$. Moreover, let $\varepsilon \in \left]0,1\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $x_0 \in \operatorname{dom} R_0$, and consider the following routine.

$$
(\forall n \in \mathbb{N}) \quad
\left\lfloor
\begin{array}{l}
y_n = R_n x_n + a_n \\
q_n = T_n y_n + b_n \\
\text{If } q_n \notin \operatorname{dom} R_n \text{ stop.} \\
\text{Else} \\
\left\lfloor
\begin{array}{l}
r_n = R_n q_n + c_n \\
z_n = x_n - y_n + r_n \\
x_{n+1} = x_n + \lambda_n \big(Q_n z_n - x_n \big)
\end{array}
\right. \\
\text{If } x_{n+1} \notin \operatorname{dom} R_{n+1} \text{ stop.} \\
\text{Else } n = n+1.
\end{array}
\right.
\tag{2.1}
$$

Our main result is the following.

Theorem 2.2 *Suppose that $Z \neq \varnothing$ in Problem 1.1 and that Algorithm 2.1 generates infinite orbits $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ such that*

$$
(\forall x \in \mathcal{H}) \quad
\begin{cases}
x_{k_n} \rightharpoonup x \\
x_n - T_n R_n x_n \to 0 \\
z_n - x_n \to 0 \\
z_n - Q_n z_n \to 0
\end{cases}
\quad \Rightarrow \quad x \in Z.
\tag{2.2}
$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a solution to Problem 1.1.

Proof. Set

$$
(\forall n \in \mathbb{N}) \quad \widetilde{y}_n = R_n x_n, \quad \widetilde{q}_n = T_n \widetilde{y}_n, \quad \text{and} \quad \widetilde{r}_n = R_n \widetilde{q}_n,
\tag{2.3}
$$

fix $z \in Z$, and let $n \in \mathbb{N}$. Note that, since $z \in S$, we have

$$
z = P_S z = Q_n z = T_n R_n z = R_n z + (\operatorname{Id} - R_n) T_n R_n z.
\tag{2.4}
$$

In addition, it follows from [9, Theorem 1] that $(\operatorname{Id} - R_n)$ is monotone, which yields $\langle (\operatorname{Id} - R_n)\widetilde{q}_n - (\operatorname{Id} - R_n) z \mid \widetilde{q}_n - z \rangle \geq 0$. Therefore, we deduce from (2.4), (2.3), and the firm nonexpansivity of T_n that

$$\begin{aligned}
2\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\rangle &= -2\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)\widetilde{q}_n - (\mathrm{Id} - R_n)z)\rangle \\
&\quad + 2\langle \widetilde{q}_n - z \mid x_n - z\rangle - 2\langle \widetilde{q}_n - z \mid R_n x_n - R_n z\rangle \\
&\leq 2\langle \widetilde{q}_n - z \mid x_n - z\rangle - 2\langle T_n\widetilde{y}_n - T_nR_nz \mid \widetilde{y}_n - R_n z\rangle \\
&\leq 2\langle \widetilde{q}_n - z \mid x_n - z\rangle - 2\|T_n\widetilde{y}_n - T_nR_nz\|^2 \\
&= \big(2\langle \widetilde{q}_n - z \mid x_n - z\rangle - \|\widetilde{q}_n - z\|^2\big) - \|\widetilde{q}_n - z\|^2 \\
&\leq \|x_n - z\|^2 - \|\widetilde{q}_n - x_n\|^2 - \|\widetilde{q}_n - z\|^2. \qquad (2.5)
\end{aligned}$$

Hence, since $\sup_{k\in\mathbb{N}} \beta_k^2 \leq (1-\varepsilon)^2 \leq 1-\varepsilon$, it follows from (2.3) and the β_n–Lipschitz property of $(\mathrm{Id} - R_n)$ that

$$\begin{aligned}
\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\|^2 &= \|\widetilde{q}_n - z + (x_n - \widetilde{y}_n) - (\widetilde{q}_n - \widetilde{r}_n)\|^2 \\
&= \|\widetilde{q}_n - z + (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\|^2 \\
&= \|\widetilde{q}_n - z\|^2 + \|(\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\|^2 \\
&\quad + 2\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\rangle \\
&\leq \|\widetilde{q}_n - z\|^2 + \beta_n^2\|\widetilde{q}_n - x_n\|^2 \\
&\quad + 2\langle \widetilde{q}_n - z \mid (\mathrm{Id} - R_n)x_n - (\mathrm{Id} - R_n)\widetilde{q}_n\rangle \\
&\leq \|x_n - z\|^2 - (1 - \beta_n^2)\|\widetilde{q}_n - x_n\|^2 \\
&\leq \|x_n - z\|^2 - \varepsilon\|\widetilde{q}_n - x_n\|^2, \qquad (2.6)
\end{aligned}$$

which yields

$$\|x_n - \widetilde{y}_n + \widetilde{r}_n - z\| \leq \|x_n - z\|. \qquad (2.7)$$

We also derive from (2.1) and (2.3) the following inequalities. First, $\|y_n - \widetilde{y}_n\| = \|a_n\|$, and since T_n is nonexpansive, we obtain

$$\|q_n - \widetilde{q}_n\| = \|T_n y_n + b_n - T_n\widetilde{y}_n\| \leq \|\widetilde{y}_n - y_n\| + \|b_n\| = \|a_n\| + \|b_n\|. \qquad (2.8)$$

In turn, it follows from the β_n–Lipschitz property of $(\mathrm{Id} - R_n)$ that

$$\begin{aligned}
\|r_n - \widetilde{r}_n\| &= \|R_n q_n + c_n - R_n\widetilde{q}_n\| \\
&\leq \|(\mathrm{Id} - R_n)\widetilde{q}_n - (\mathrm{Id} - R_n)q_n\| + \|q_n - \widetilde{q}_n\| + \|c_n\| \\
&\leq (1 + \beta_n)\|q_n - \widetilde{q}_n\| + \|c_n\| \\
&\leq 2(\|a_n\| + \|b_n\|) + \|c_n\|. \qquad (2.9)
\end{aligned}$$

Altogether, if we set

$$e_n = \widetilde{y}_n - y_n + r_n - \widetilde{r}_n, \qquad (2.10)$$

we have

$$\|e_n\| = \|\widetilde{y}_n - y_n + r_n - \widetilde{r}_n\| \leq \|y_n - \widetilde{y}_n\| + \|r_n - \widetilde{r}_n\| \leq 3\|a_n\| + 2\|b_n\| + \|c_n\|, \qquad (2.11)$$

and therefore $\sum_{k\in\mathbb{N}} \|e_k\| < +\infty$. Hence, from (2.1), (2.4), the nonexpansivity of Q_n, and (2.7) we get

$$\begin{aligned}\|x_{n+1}-z\| &= \|(1-\lambda_n)(x_n-z)+\lambda_n(Q_nz_n-Q_nz)\|\\ &\leq (1-\lambda_n)\|x_n-z\|+\lambda_n\|Q_nz_n-Q_nz\|\\ &\leq (1-\lambda_n)\|x_n-z\|+\lambda_n\|z_n-z\|\\ &\leq (1-\lambda_n)\|x_n-z\|+\lambda_n\big(\|x_n-\widetilde{y}_n+\widetilde{r}_n-z\|+\|e_n\|\big)\\ &\leq \|x_n-z\|+\|e_n\|,\end{aligned} \tag{2.12}$$

and we conclude from [10, Lemma 3.1] that

$$\xi = \sup_{k\in\mathbb{N}} \|x_k - z\| < +\infty. \tag{2.13}$$

Thus, from the convexity of $\|\cdot\|^2$, the firm nonexpansivity of Q_n, (2.4), (2.1), (2.10), (2.6), and (2.7) we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\leq (1-\lambda_n)\|x_n-z\|^2+\lambda_n\|Q_nz_n-Q_nz\|^2\\ &\leq (1-\lambda_n)\|x_n-z\|^2+\lambda_n\big(\|z_n-z\|^2-\|z_n-Q_nz_n\|^2\big)\\ &\leq (1-\lambda_n)\|x_n-z\|^2+\lambda_n\big(\|x_n-\widetilde{y}_n+\widetilde{r}_n-z\|^2+\|e_n\|^2\\ &\qquad\qquad +2\,\|x_n-\widetilde{y}_n+\widetilde{r}_n-z\|\,\|e_n\|-\|z_n-Q_nz_n\|^2\big)\\ &\leq (1-\lambda_n)\|x_n-z\|^2+\lambda_n\big(\|x_n-z\|^2-\varepsilon\|\widetilde{q}_n-x_n\|^2\\ &\qquad\qquad +\|e_n\|^2+2\,\|x_n-z\|\,\|e_n\|-\|z_n-Q_nz_n\|^2\big)\\ &\leq \|x_n-z\|^2-\varepsilon^2\|\widetilde{q}_n-x_n\|^2-\varepsilon\,\|z_n-Q_nz_n\|^2+\eta_n,\end{aligned} \tag{2.14}$$

where $\eta_n = \|e_n\|^2 + 2\xi\|e_n\|$ satisfies $\sum_{k\in\mathbb{N}} \eta_k < +\infty$. Hence, from [10, Lemma 3.1] we deduce that

$$\sum_{k\in\mathbb{N}} \|T_kR_kx_k - x_k\|^2 = \sum_{k\in\mathbb{N}} \|\widetilde{q}_k - x_k\|^2 < +\infty \quad \text{and} \quad \sum_{k\in\mathbb{N}} \|z_k - Q_kz_k\|^2 < +\infty, \tag{2.15}$$

and therefore $T_nR_nx_n - x_n = \widetilde{q}_n - x_n \to 0$ and $z_n - Q_nz_n \to 0$. Thus, it follows from (2.1) and the nonexpansivity of T_n that

$$\begin{aligned}\|z_n-x_n\| &= \|r_n-y_n\|\\ &= \|\widetilde{r}_n-\widetilde{y}_n+e_n\|\\ &\leq \|T_n\widetilde{q}_n-T_nx_n\|+\|e_n\|\\ &\leq \|\widetilde{q}_n-x_n\|+\|e_n\|\\ &\to 0.\end{aligned} \tag{2.16}$$

Altogether, since (2.2) asserts that all the weak limits of the sequence $(x_k)_{k\in\mathbb{N}}$ are in Z, the result follows from [10, Theorem 3.8]. □

3 Monotone inclusions with convex constraints

We consider the problem

$$\text{find} \quad x \in S \quad \text{such that} \quad 0 \in Ax + Bx, \tag{3.1}$$

where $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ are maximally monotone, and $S \subset \mathcal{H}$ is nonempty, closed, and convex. When B is cocoercive, $\operatorname{dom} B = \mathcal{H}$, and $S = \mathcal{H}$, (3.1) models wide variety of problems in nonlinear analysis, and it can be solved by the forward-backward splitting method [11, 12, 13, 14, 15, 16]. However, in several applications these assumptions are very restrictive. If the cocoercivity of B is relaxed to a Lipschitz property, (3.1) can be solved by the modified forward-backward splitting in [8]. We propose an extension of this method for solving (3.1) with a finite number of convex constraints. In addition, our method allows for errors in the computations of the operators involved.

Notation 3.1 For a set-valued operator $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, $\operatorname{dom} A = \{x \in \mathcal{H} \mid Ax \neq \varnothing\}$ is the domain of A, $\operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$ is its set of zeros, and $\operatorname{gra} A = \{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax\}$ is its graph. The operator A is monotone if it satisfies, for every (x,u) and (y,v) in $\operatorname{gra} A$, $\langle x-y \mid u-v\rangle \geq 0$, and it is maximally monotone if its graph is not properly contained in the graph of any other monotone operator acting on $\mathcal{H}$. In this case, the resolvent of A, $J_A = (\operatorname{Id} + A)^{-1}$, is well defined, single-valued, $\operatorname{dom} J_A = \mathcal{H}$, and it is firmly nonexpansive. For every $\alpha \in \mathbb{R}$, the lower level set at height α of a function $f\colon \mathcal{H} \to \left]-\infty,+\infty\right]$ is the closed convex set $\operatorname{lev}_{\leq \alpha} f = \{x \in \mathcal{H} \mid f(x) \leq \alpha\}$ and the subdifferential of f is the operator

$$\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H}) \quad \langle y-x \mid u\rangle + f(x) \leq f(y)\big\}. \tag{3.2}$$

Now let C be a nonempty subset of $\mathcal{H}$. Then $\operatorname{int} C$ is the interior of C and if C is nonempty, convex, and closed, then P_C denotes the projector operator onto C, which, for every $x \in \mathcal{H}$ satisfies $\|x - P_C x\| = \min_{y\in C} \|x-y\| = d_C(x)$, where d_C denotes the distance function of C. For further background in monotone operator theory and convex analysis see [17].

Problem 3.2 Let $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ and $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ be two maximally monotone operators such that $\operatorname{dom} A \subset \operatorname{dom} B$ and suppose that $A+B$ is maximally monotone (see [17, Corollary 24.4] for some sufficient conditions). For every $i \in \{1,\ldots,m\}$, let $f_i\colon \mathcal{H} \to \mathbb{R}$ be lower semicontinuous and convex, denote by $S = \operatorname{lev}_{\leq 0} f_1 \cap \cdots \cap \operatorname{lev}_{\leq 0} f_m \neq \varnothing$, and assume that $S \subset \operatorname{dom} B$ and that B is χ-Lipschitzian on $S \cup \operatorname{dom} A$, for some $\chi \in \left]0,+\infty\right[$. The problem is to

$$\text{find} \quad x \in \mathcal{H} \quad \text{such that} \quad \begin{cases} x \in \operatorname{zer}(A+B) \\ f_1(x) \leq 0 \\ \quad\vdots \\ f_m(x) \leq 0. \end{cases} \tag{3.3}$$

Problem 3.2 models various applications to economics, traffic theory, Nash equilibrium problems, and network equilibrium problems among others (see [18, 19, 20] and the references therein).

In the particular case when $m=1$, $f_1 = d_C$, and $C \subset \mathcal{H}$ is a nonempty closed convex set, an algorithm for solving Problem 3.2 is proposed in [8], without considering errors in the computations and assuming that P_C is easily computable (see also [21] for an approach using enlargements of maximally monotone operators). However, since P_S is not computable in general, Problem 3.2 can not be solved by this method. We propose an algorithm for solving Problem 3.2 in which the constraints $f_1 \leq 0, \ldots, f_m \leq 0$ are activated independently and linearized, and where errors in the computation of the operators involved are permitted. For the implementation of this method we use the subgradient projector with respect to $f \in \Gamma_0(\mathcal{H})$, which is defined by

$$G\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto \begin{cases} x - \dfrac{f(x)}{\|u\|^2}u, & \text{if } f(x) > 0; \\ x, & \text{otherwise}, \end{cases} \tag{3.4}$$

where $u \in \partial f(x)$, and the function $\mathrm{i}\colon \mathbb{N} \to \{1,\ldots,m\}\colon n \mapsto 1+\mathrm{rem}(n-1,m)$, where $\mathrm{rem}(\cdot,m)$ is the remainder function of division by m.

Algorithm 3.3 In Problem 3.2, for every $i \in \{1,\ldots,m\}$, denote by $G_i\colon \mathcal{H} \to \mathcal{H}$ the subgradient projector with respect to f_i. Let $(e_{1,n})_{n\in\mathbb{N}}$, $(e_{2,n})_{n\in\mathbb{N}}$, and $(e_{3,n})_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$, $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$, and $\sum_{n\in\mathbb{N}} \|e_{3,n}\| < +\infty$. Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$, let $x_0 \in \operatorname{dom} B$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n(Bx_n + e_{1,n}) \\ q_n = J_{\gamma_n A}(y_n + e_{2,n}) \\ r_n = q_n - \gamma_n(Bq_n + e_{3,n}) \\ z_n = x_n - y_n + r_n \\ x_{n+1} = G_{\mathrm{i}(n)}\, z_n. \end{array} \right. \tag{3.5}$$

Remark 3.4 In Algorithm 3.3, the sequences $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{3,n})_{n\in\mathbb{N}}$ represent errors in the computation of the operator B. In addition, we suppose that the resolvents $(J_{\gamma_n A})_{n\in\mathbb{N}}$ can be computed approximatively by solving, for every $n \in \mathbb{N}$, the perturbed inclusion

$$\text{find} \quad q \in \mathcal{H} \quad \text{such that} \quad y_n - q + e_{2,n} \in \gamma_n A q. \tag{3.6}$$

Proposition 3.5 *Suppose that*

$$\bigcup_{i=1}^{m} \operatorname{ran} G_i \subset \operatorname{dom} B \quad \textit{and} \quad S \cap \operatorname{zer}(A+B) \neq \varnothing. \tag{3.7}$$

Then Algorithm 3.3 generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$ which converges weakly to a solution to Problem 3.2.

Proof. Set

$$(\forall n \in \mathbb{N}) \quad \beta_n = \gamma_n\chi, \quad T_n = J_{\gamma_n A}, \quad \text{and} \quad R_n = \mathrm{Id} - \gamma_n B. \tag{3.8}$$

Note that $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $]0, 1-\varepsilon]$ and, for every $n \in \mathbb{N}$, T_n is firmly nonexpansive and $\mathrm{Id} - R_n = \gamma_n B$ is β_n–Lipschitzian and monotone. Hence, it follows from [9, Theorem 1] that the operators $(R_n)_{n\in\mathbb{N}}$ are pseudo contractive. In addition, note that $x \in \operatorname{zer}(A+B) \Leftrightarrow (\forall n \in \mathbb{N}) \quad x - \gamma_n Bx \in x + \gamma_n Ax \Leftrightarrow (\forall n \in \mathbb{N}) \quad x \in \operatorname{Fix} T_n R_n$. Altogether, we deduce that Problem 3.2 is a particular case of Problem 1.1 and

$$Z = S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n R_n = S \cap \operatorname{zer}(A+B) \neq \varnothing. \tag{3.9}$$

Now let us prove that Algorithm 3.3 is a particular case of Algorithm 2.1. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} a_n = -\gamma_n e_{1,n} \\ b_n = J_{\gamma_n A}(y_n + e_{2,n}) - J_{\gamma_n A} y_n \\ c_n = -\gamma_n e_{3,n} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_n = 1 \\ Q_n = G_{\mathrm{i}(n)}. \end{cases} \tag{3.10}$$

Then, since $\sup_{n\in\mathbb{N}} \gamma_n < \chi^{-1}$, we have $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|c_n\| < +\infty$. Moreover, from the nonexpansivity of $(J_{\gamma_n A})_{n\in\mathbb{N}}$, we deduce that $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$, and, for every $x \in \mathcal{H}$ and $n \in \mathbb{N}$, $Q_n x$ is the projection onto the closed affine half-space $\big\{y \in \mathcal{H} \;\big|\; \langle x - y \mid u\rangle \geq f_{\mathrm{i}(n)}(x)\big\}$, for some $u \in \partial f_{\mathrm{i}(n)}(x)$, which contains $\mathrm{lev}_{\leq 0} f_{\mathrm{i}(n)} \supset S$. On the other hand, $x_0 \in \mathrm{dom}\, B$ and since, for every $i \in \{1, \ldots, m\}$, $\mathrm{ran}\, G_i \subset \mathrm{dom}\, B$, it follows from (3.5) that, for every $n \in \mathbb{N} \setminus \{0\}$, $x_n \in \mathrm{dom}\, B$. In addition, $q_n = J_{\gamma_n A}(y_n + e_{2,n}) \in \mathrm{dom}\, A \subset \mathrm{dom}\, B$. Altogether, from (3.8) and (3.10), we deduce that Algorithm 3.3 is a particular case of Algorithm 2.1 and that it generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$.

Let us prove that condition (2.2) holds. Suppose that $x_{k_n} \rightharpoonup x$, $x_n - T_n R_n x_n \to 0$, $z_n - x_n \to 0$, $z_n - Q_n z_n \to 0$, and, for every $n \in \mathbb{N}$, denote by $p_n = T_n R_n x_n$. Hence, $p_{k_n} \rightharpoonup x$ and from (3.8) we obtain, for every $n \in \mathbb{N}$,

$$\begin{aligned} p_n = T_n R_n x_n \quad &\Leftrightarrow \quad x_n - \gamma_n B x_n \in p_n + \gamma_n A p_n \\ &\Leftrightarrow \quad \frac{1}{\gamma_n}(x_n - p_n) - B x_n \in A p_n \\ &\Leftrightarrow \quad \frac{1}{\gamma_n}(x_n - p_n) + B p_n - B x_n \in (A + B) p_n. \end{aligned} \tag{3.11}$$

Now, since $A + B$ is maximally monotone, from [17, Proposition 20.33], its graph is sequentially weak-strong closed. Therefore, since $x_{k_n} - p_{k_n} \to 0$, $\|B p_{k_n} - B x_{k_n}\| \leq \chi \|x_{k_n} - p_{k_n}\| \to 0$, $\gamma_{k_n} \geq \varepsilon > 0$, $p_{k_n} \rightharpoonup x$, we conclude from (3.11) that $x \in \mathrm{zer}(A + B)$. Now let us prove that, for every $i \in \{1, \ldots, m\}$, $f_i(x) \leq 0$. Fix $i \in \{1, \ldots, m\}$ and, for every $n \in \mathbb{N}$, let $j_n \in \mathbb{N}$ such that $k_n \leq j_n \leq k_n + m$ and $\mathrm{i}(j_n) = i$. We deduce from $z_n - x_n \to 0$ and $z_n - Q_n z_n \to 0$ that, for every $n \in \mathbb{N}$, $\|x_{n+1} - x_n\| = \|Q_n z_n - x_n\| \leq \|Q_n z_n - z_n\| + \|z_n - x_n\| \to 0$. Therefore,

$$(\forall n \in \mathbb{N}) \quad \|x_{j_n} - x_{k_n}\| \leq \sum_{\ell=k_n}^{j_n - 1} \|x_{\ell+1} - x_\ell\| \leq m \max_{k_n \leq \ell \leq k_n + m} \|x_{\ell+1} - x_\ell\| \to 0 \tag{3.12}$$

and hence it follows from $z_{j_n} - x_{j_n} \to 0$ and $x_{k_n} \rightharpoonup x$ that $z_{j_n} \rightharpoonup x$. Note that, from (3.10) and (3.4) we have, for some $u_{j_n} \in \partial f_i(z_{j_n})$,

$$(\forall n \in \mathbb{N}) \quad Q_{j_n} z_{j_n} - z_{j_n} = \begin{cases} -\dfrac{f_i(z_{j_n})}{\|u_{j_n}\|^2} u_{j_n}, & \text{if } f_i(z_{j_n}) > 0; \\ 0, & \text{otherwise,} \end{cases} \tag{3.13}$$

and, since $\|Q_{j_n} z_{j_n} - z_{j_n}\| \to 0$, we deduce that $\max\{0, f_i(z_{j_n})\} \to 0$. Thus, it follows from $z_{j_n} \rightharpoonup x$ that $f_i(x) \leq \underline{\lim}\, f_i(z_{j_n}) \leq \underline{\lim}\, \max\{0, f_i(z_{j_n})\} = 0$, and hence $x \in \mathrm{lev}_{\leq 0} f_i$. We conclude that $x \in Z$ and the result follows from Theorem 2.2. ◻

Remark 3.6 Let us consider the particular case of Theorem 3.5 obtained when $e_{1,n} \equiv e_{2,n} \equiv e_{3,n} \equiv 0$, $m = 1$, and $f_1 = d_C$, where $C \subset \mathcal{H}$ is a nonempty closed convex set. Then, since $G_1 = P_C$, Algorithm 3.3 reduces to the method proposed in [8]. Moreover, since $S = C$, note that the assumption $\mathrm{ran}\, G_1 \subset \mathrm{dom}\, B$ is equivalent to $S \subset \mathrm{dom}\, B$, which was already assumed in Problem 3.2.

4 Equilibrium problems with convex constraints

We consider the problem

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F(x,y) \geq 0, \tag{4.1}$$

where C and F satisfy the following assumption.

Assumption 4.1 C is a nonempty closed convex subset of $\mathcal{H}$ and $F\colon C^2 \to \mathbb{R}$ satisfies the following.

(i) $(\forall x \in C) \quad F(x,x) = 0$.

(ii) $(\forall (x,y) \in C^2) \quad F(x,y) + F(y,x) \leq 0$.

(iii) For every x in C, $F(x,\cdot)\colon C \to \mathbb{R}$ is lower semicontinuous and convex.

(iv) $(\forall (x,y,z) \in C^3) \quad \varlimsup_{\varepsilon \to 0^+} F((1-\varepsilon)x + \varepsilon z, y) \leq F(x,y)$.

We are interested in solving a more general problem than (4.1), which involves a finite or a countable infinite number of convex constraints. It will be presented after the following preliminaries.

Notation 4.2 The resolvent of $F\colon C^2 \to \mathbb{R}$ is the set valued operator

$$J_F\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{ z \in C \;\big|\; (\forall y \in C) \quad F(z,y) + \langle y - z \mid z - x \rangle \geq 0 \big\} \tag{4.2}$$

and, for every $\delta \in \left]0,+\infty\right[$, the δ–resolvent of $F\colon C^2 \to \mathbb{R}$ is the set valued operator

$$J_F^\delta\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{ z \in C \;\big|\; (\forall y \in C) \quad F(z,y) + \langle y - z \mid z - x \rangle \geq -\delta \big\}. \tag{4.3}$$

Lemma 4.3 *Let F and C be such that Assumption 4.1 holds. Then the following hold.*

(i) $\operatorname{dom} J_F = \mathcal{H}$.

(ii) *J_F is single-valued and firmly nonexpansive.*

(iii) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F x \in J_F^\delta x$.

(iv) $(\forall x \in \mathcal{H})(\forall \delta \in \left]0,+\infty\right[) \quad J_F^\delta x \subset B(J_F x; \sqrt{\delta})$.

Proof. (i)&(ii): [2, Lemma 2.12]. (iii): This follows from (ii), (4.2), and (4.3). (iv): Fix $x \in \mathcal{H}$ and $\delta \in \left]0,+\infty\right[$, and let $w \in J_F^\delta x$. We deduce from (4.2) and (4.3) that $F(J_F x, w) + \langle w - J_F x \mid J_F x - x \rangle \geq 0$ and $F(w, J_F x) + \langle J_F x - w \mid w - x \rangle \geq -\delta$, respectively. Adding both inequalities we obtain $F(w, J_F x) + F(J_F x, w) - \|J_F x - w\|^2 \geq -\delta$. Hence, it follows from Assumption (ii) that $\|J_F x - w\|^2 \leq \delta$, which yields the result. ◻

Problem 4.4 Let F and C be such that Assumption 4.1 holds. Let $(S_i)_{i\in I}$ be a countable (finite or countable infinite) family of closed convex subsets of $\mathcal{H}$ such that $S = \cap_{i\in I} S_i \neq \varnothing$. Let $B\colon \operatorname{dom} B \subset \mathcal{H} \to \mathcal{H}$ be a monotone and χ–Lipschitzian operator for some $\chi \in \left]0,+\infty\right[$ such that $C \subset \operatorname{dom} B$, and suppose that

$$\bigcup_{i\in I} S_i \subset \operatorname{int}\operatorname{dom} B. \tag{4.4}$$

The problem is to

$$\text{find}\quad x \in S \quad \text{such that} \quad (\forall y \in C) \quad F(x,y) + \langle y - x \mid Bx \rangle \geq 0. \tag{4.5}$$

Problem 4.4 models a wide variety of problems including complementarity problems, optimization problems, feasibility problems, Nash equilibrium problems, variational inequalities, and fixed point problems [10, 2, 22, 23, 24, 25].

In the literature, there exist some splitting algorithms for solving the equilibrium problem

$$\text{find}\quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F_1(x,y) + F_2(x,y) \geq 0, \tag{4.6}$$

where F_1 and F_2 satisfy Assumption 4.1. These methods take advantage of the properties of F_1 and F_2 separately. For instance, sequential and parallel splitting algorithms are proposed in [26], where the resolvents J_{F_1} and J_{F_2} are used. The ergodic convergence to a solution to (4.6) is established without additional assumptions. However, when $F_1 = F$ and $F_2\colon (x,y) \mapsto \langle y - x \mid Bx \rangle$ we have $J_{F_2} = J_B = (\operatorname{Id} + B)^{-1}$ [2, Lemma 2.15(i)], which is often difficult to compute, even in the linear case. Moreover, the ergodic method proposed in [26] involves vanishing parameters that leads to numerical instabilities, which make it of limited use in applications. In [2, 27] a different approach is developed to overcome this disadvantage when B is cocoercive. In their methods, the operator B is computed explicitly and the weakly convergence to a solution to (4.5) when $S = C$ is demonstrated.

In this section we propose the following non-ergodic algorithm for solving the general case considered in Problem 4.4. This approach can deal with errors in the computations of the operators involved. The convergence of the proposed method is a consequence of Theorem 2.2.

Algorithm 4.5 In Problem 4.4 let $(I_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of I, let $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{2,n})_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \|e_{1,n}\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|e_{2,n}\| < +\infty$, and let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0,+\infty\right[$ such that $\sum_{n\in\mathbb{N}} \sqrt{\delta_n} < +\infty$. Let $\varepsilon \in \left]0, 1/(\chi+1)\right[$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (1-\varepsilon)/\chi]$, let $\cup_{n\in\mathbb{N}}\{\omega_{i,n}\}_{i\in I_n} \subset [\varepsilon, 1]$ be such that, for every $n \in \mathbb{N}$, $\sum_{i\in I_n} \omega_{i,n} = 1$, let $x_0 \in \operatorname{dom} B$, and let $(x_n)_{n\in\mathbb{N}}$ be the sequence generated by the following routine.

$$(\forall n \in \mathbb{N}) \quad \left\lfloor \begin{array}{l} y_n = x_n - \gamma_n (B x_n + e_{1,n}) \\ q_n \in J^{\delta_n}_{\gamma_n F} y_n \\ r_n = q_n - \gamma_n (B q_n + e_{2,n}) \\ z_n = x_n - y_n + r_n \\ x_{n+1} = \sum_{i\in I_n} \omega_{i,n} P_{S_i} z_n. \end{array} \right. \tag{4.7}$$

Remark 4.6 In Algorithm 4.5, the sequences $(e_{1,n})_{n\in\mathbb{N}}$ and $(e_{2,n})_{n\in\mathbb{N}}$ represent errors in the computation of the operator B. On the other hand, it follows from (4.7) and (4.3) that, for every $n \in \mathbb{N}$, q_n is a solution to

$$\text{find} \quad q \in C \quad \text{such that} \quad (\forall y \in C) \quad F(q, y) + \langle y - y_n \mid y - q\rangle \geq -\delta_n. \tag{4.8}$$

Thus, we obtain from (4.2) that q_n can be interpreted as an approximate computation of the resolvent $J_{\gamma_n F} y_n$.

Proposition 4.7 *Suppose that there exist strictly positive integers $(M_i)_{i\in I}$ and N such that*

$$(\forall (i,n) \in I \times \mathbb{N}) \quad i \in \bigcup_{k=n}^{n+M_i-1} I_k \quad \text{and} \quad 1 \leq \operatorname{card} I_n \leq N, \tag{4.9}$$

and that Problem 4.4 admits at least one solution. Then Algorithm 4.5 generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$ which converges weakly to a solution to Problem 4.4.

Proof. First, let us prove that Problem 4.4 is a particular case of Problem 1.1. Set

$$(\forall n \in \mathbb{N}) \quad \beta_n = \gamma_n \chi, \ T_n = J_{\gamma_n F}, \ \text{and} \ R_n = \operatorname{Id} - \gamma_n B. \tag{4.10}$$

Note that $(\beta_n)_{n\in\mathbb{N}}$ is a sequence in $]0, 1-\varepsilon]$ and, for every $n \in \mathbb{N}$, T_n is firmly nonexpansive [2, Lemma 2.12] and $\operatorname{Id} - R_n = \gamma_n B$ is β_n–Lipschitzian and monotone. Hence, it follows from [9, Theorem 1] that the operators $(R_n)_{n\in\mathbb{N}}$ are pseudo contractive. In addition, we deduce from (4.2) and (4.10) that $(\forall n \in \mathbb{N}) \quad x \in \operatorname{Fix} T_n R_n \ \Leftrightarrow \ (\forall n \in \mathbb{N})(\forall y \in C)\ \gamma_n F(x, y) + \langle y - x \mid x - R_n x\rangle \geq 0 \ \Leftrightarrow \ (\forall y \in C)\ F(x, y) + \langle y - x \mid Bx\rangle \geq 0$. Altogether, we deduce that Problem 4.4 is a particular case of Problem 1.1 and

$$Z = S \cap \bigcap_{n\in\mathbb{N}} \operatorname{Fix} T_n R_n = S \cap \big\{x \in C \ \big|\ (\forall y \in C)\ F(x, y) + \langle y - x \mid Bx\rangle \geq 0\big\} \neq \varnothing. \tag{4.11}$$

Now let us show that Algorithm 4.5 is deduced from Algorithm 2.1. Set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} a_n = -\gamma_n e_{1,n} \\ b_n = q_n - J_{\gamma_n F} y_n \\ c_n = -\gamma_n e_{2,n} \end{cases} \quad \text{and} \quad \begin{cases} \lambda_n = 1 \\ Q_n = \sum_{i\in I_n} \omega_{i,n} P_{S_i}. \end{cases} \tag{4.12}$$

Then, since $\sup_{n\in\mathbb{N}} \gamma_n < \chi^{-1}$, we have $\sum_{n\in\mathbb{N}} \|a_n\| < +\infty$ and $\sum_{n\in\mathbb{N}} \|c_n\| < +\infty$. Moreover, it follows from (4.7) and Lemma (iv) that $\sum_{n\in\mathbb{N}} \|b_n\| < +\infty$, and, for every $x \in \mathcal{H}$ and $n \in \mathbb{N}$, $Q_n x$ is the projection onto the closed affine half-space $H_n(x) = \big\{z \in \mathcal{H} \ \big|\ \langle z - Q_n x \mid x - Q_n x\rangle \leq 0\big\}$, which satisfies $S \subset \cap_{i\in I_n} S_i = \operatorname{Fix} Q_n \subset H_n(x)$ [10, Proposition 2.4]. On the other hand, we have $x_0 \in \operatorname{dom} B$ and it follows from (4.4) and the convexity of $\operatorname{int} \operatorname{dom} B$ [28, Theorem 27.1] that

$$(\forall n \in \mathbb{N}) \quad \operatorname{ran}\left(\sum_{i\in I_n} \omega_{i,n} P_{S_i}\right) \subset \operatorname{conv}\left(\bigcup_{i\in I_n} S_i\right) \subset \operatorname{conv}\left(\bigcup_{i\in I} S_i\right) \subset \operatorname{int} \operatorname{dom} B. \tag{4.13}$$

Hence, we conclude from (4.7) that, for every $n \in \mathbb{N}\backslash\{0\}$, $x_n \in \operatorname{int}\operatorname{dom} B$. Moreover, for every $n \in \mathbb{N}$, $q_n \in C \subset \operatorname{dom} B$. Altogether, from (4.10) and (4.12), we deduce that Algorithm 4.5 is a particular case of Algorithm 2.1, which generates an infinite orbit $(x_n)_{n\in\mathbb{N}}$.

Finally, let us show that (2.2) holds. Suppose that $x_{k_n} \rightharpoonup x$, $x_n - T_n R_n x_n \to 0$, $z_n - x_n \to 0$, $z_n - Q_n z_n \to 0$, and, for every $n \in \mathbb{N}$, denote by $p_n = T_n R_n x_n$. Hence, $p_{k_n} \rightharpoonup x$ and it follows from (4.10) and (4.2) that, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
p_n = T_n R_n x_n \;\Leftrightarrow\;& (\forall z \in C)\;\; F(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle + \langle z - p_n \mid Bx_n\rangle \geq 0 \\
\Leftrightarrow\;& (\forall z \in C)\;\; F(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle \\
&\qquad\qquad + \langle z - p_n \mid Bx_n - Bp_n\rangle + \langle z - p_n \mid Bp_n\rangle \geq 0 \\
\Leftrightarrow\;& (\forall z \in C)\;\; G(p_n, z) + \frac{1}{\gamma_n}\langle z - p_n \mid p_n - x_n\rangle + \langle z - p_n \mid Bx_n - Bp_n\rangle \geq 0,
\end{aligned}
\tag{4.14}
$$

where

$$
G\colon C^2 \to \mathbb{R}\colon (x, y) \mapsto F(x, y) + \langle y - x \mid Bx\rangle \tag{4.15}
$$

satisfies the Assumption 4.1 [2, Lemma 2.15(i)]. Moreover, since $\inf_{n\in\mathbb{N}} \gamma_{k_n} > 0$, $x_{k_n} - p_{k_n} \to 0$, and $(p_{k_n})_{n\in\mathbb{N}}$ is bounded, we have $(\forall z \in C)\; \langle z - p_{k_n} \mid p_{k_n} - x_{k_n}\rangle/\gamma_{k_n} \to 0$, and from the Lipschitz property of B we obtain $(\forall z \in C)\; \langle z - p_{k_n} \mid Bx_{k_n} - Bp_{k_n}\rangle \to 0$. Hence, we deduce from $p_{k_n} \rightharpoonup x$, Assumption 4.1(iii), Assumption 4.1(ii), and (4.14) that

$$
\begin{aligned}
(\forall z \in C) \quad G(z, x) &\leq \underline{\lim}\, G(z, p_{k_n}) \\
&\leq \underline{\lim}\, -G(p_{k_n}, z) \\
&\leq \underline{\lim}\, \frac{1}{\gamma_{k_n}}\langle z - p_{k_n} \mid p_{k_n} - x_{k_n}\rangle + \langle z - p_{k_n} \mid Bx_{k_n} - Bp_{k_n}\rangle \\
&= 0.
\end{aligned}
\tag{4.16}
$$

Now let $\varepsilon \in \left]0, 1\right]$ and $y \in C$. By convexity of C we have $x_\varepsilon = (1-\varepsilon)x + \varepsilon y \in C$. Thus, Assumption 4.1(i), Assumption 4.1(iii), and (4.16) with $z = x_\varepsilon$ yield

$$
0 = G(x_\varepsilon, x_\varepsilon) \leq (1-\varepsilon)G(x_\varepsilon, x) + \varepsilon G(x_\varepsilon, y) \leq \varepsilon G(x_\varepsilon, y), \tag{4.17}
$$

whence $G(x_\varepsilon, y) \geq 0$. In view of Assumption 4.1(iv), we conclude that $G(x, y) \geq \overline{\lim}_{\varepsilon\to 0^+} G(x_\varepsilon, y) \geq 0$, which yields

$$
(\forall y \in C) \quad G(x, y) = F(x, y) + \langle y - x \mid Bx\rangle \geq 0. \tag{4.18}
$$

Now, let us prove that $x \in S$. Since $z_n - x_n \to 0$ and $z_n - Q_n z_n \to 0$, (4.12) yields

$$
(\forall n \in \mathbb{N}) \quad \|x_{n+1} - x_n\| = \|Q_n z_n - x_n\| \leq \|Q_n z_n - z_n\| + \|z_n - x_n\| \to 0. \tag{4.19}
$$

Now, fix $i \in I$. In view of (4.9), there exists a sequence $(j_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that, for every $n \in \mathbb{N}$, $k_n \leq j_n \leq k_n + M_i - 1$ and $i \in I_{j_n}$. For every $n \in \mathbb{N}$, it follows from (4.19) that

$$
\|x_{j_n} - x_{k_n}\| \leq \sum_{\ell=k_n}^{k_n+M_i-2} \|x_{\ell+1} - x_\ell\| \leq (M_i - 1) \max_{k_n \leq \ell \leq k_n+M_i-2} \|x_{\ell+1} - x_\ell\| \to 0. \tag{4.20}
$$

Thus, we deduce from $x_{k_n} \rightharpoonup x$ and $z_{j_n} - x_{j_n} \to 0$ that $z_{j_n} \rightharpoonup x$. On the other hand, let $z \in S$ and $n \in \mathbb{N}$. Since, for every $\ell \in I_{j_n}$, $P_{S_\ell} z = z$, and $\mathrm{Id} - P_{S_\ell}$ is firmly nonexpansive, from (4.7) and (4.12) we have

$$\begin{aligned}
\|P_{S_i} z_{j_n} - z_{j_n}\|^2 &\leq \max_{\ell \in I_{j_n}} \|P_{S_\ell} z_{j_n} - z_{j_n}\|^2 \\
&\leq \frac{1}{\varepsilon} \sum_{\ell \in I_{j_n}} \omega_{\ell, j_n} \|P_{S_\ell} z_{j_n} - z_{j_n}\|^2 \\
&\leq \frac{1}{\varepsilon} \sum_{\ell \in I_{j_n}} \omega_{\ell, j_n} \langle z - z_{j_n} \mid (\mathrm{Id} - P_{S_\ell}) z - (\mathrm{Id} - P_{S_\ell}) z_{j_n} \rangle \\
&= \frac{1}{\varepsilon} \left\langle z - z_{j_n} \,\middle|\, \sum_{\ell \in I_{j_n}} \omega_{\ell, j_n} P_{S_\ell} z_{j_n} - z_{j_n} \right\rangle \\
&\leq \frac{1}{\varepsilon} \|z - z_{j_n}\| \, \|Q_{j_n} z_{j_n} - z_{j_n}\|.
\end{aligned} \tag{4.21}$$

Hence, since $(z_{j_n})_{n \in \mathbb{N}}$ is a bounded sequence and $Q_{j_n} z_{j_n} - z_{j_n} \to 0$, we deduce that $P_{S_i} z_{j_n} - z_{j_n} \to 0$. The maximally monotonicity of $\mathrm{Id} - P_{S_i}$ yields that its graph is sequentially weakly-strongly closed, and since $z_{j_n} \rightharpoonup x$, we conclude that $x = P_{S_i} x \in S_i$. Altogether, from (4.18) and (4.11) we deduce that $x \in Z$, and the result follows from Theorem 2.2. ☐

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