A DOUGLAS–RACHFORD SPLITTING METHOD FOR SOLVING EQUILIBRIUM PROBLEMS☆

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Abstract

We propose a splitting method for solving equilibrium problems involving the sum of two bifunctions satisfying standard conditions. We prove that this problem is equivalent to find a zero of the sum of two appropriate maximally monotone operators under a suitable qualification condition. Our algorithm is a consequence of the Douglas–Rachford splitting applied to this auxiliary monotone inclusion. Connections between monotone inclusions and equilibrium problems are studied.

Keywords: Douglas–Rachford splitting, equilibrium problem, maximally monotone operator, monotone inclusion
2010 MSC: Primary: 90B74, 90C30. Secondary: 47J20, 47J22

1. Introduction

In the past years, several works have been devoted to the equilibrium problem

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad H(x,y) \geq 0, \tag{1.1}$$

where C is a nonempty closed convex subset of the real Hilbert space $\mathcal{H}$, and $H\colon C \times C \to \mathbb{R}$ satisfies the following assumption.

Assumption 1.1 The bifunction $H\colon C \times C \to \mathbb{R}$ satisfies

☆This work was supported by CONICYT under grant FONDECYT N° 3120054.
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(i) $(\forall x \in C) \quad H(x,x) = 0$.
(ii) $(\forall (x,y) \in C \times C) \quad H(x,y) + H(y,x) \leq 0$.
(iii) For every x in C, $H(x,\cdot)\colon C \to \mathbb{R}$ is lower semicontinuous and convex.
(iv) $(\forall (x,y,z) \in C^3) \quad \varlimsup_{\varepsilon \to 0^+} H((1-\varepsilon)x + \varepsilon z, y) \leq H(x,y)$.

Throughout this paper, the solution set of (1.1) will be denoted by S_H.

Problem (1.1) models a wide variety of problems including complementarity problems, optimization problems, feasibility problems, Nash equilibrium problems, variational inequalities, and fixed point problems [1, 4, 5, 11, 16, 17, 18, 20, 24, 29]. Sometimes the bifunction H is difficult to manipulate but it can be considered as the sum of two simpler bifunctions F and G satisfying Assumption 1.1 (see, for example, [27]). This is the context in which we aim to solve problem (1.1). Our problem is formulated as follows.

Problem 1.2 Let C be a nonempty closed convex subset of the real Hilbert space $\mathcal{H}$. Suppose that $F\colon C \times C \to \mathbb{R}$ and $G\colon C \times C \to \mathbb{R}$ are two bifunctions satisfying Assumption 1.1. The problem is to

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F(x,y) + G(x,y) \geq 0, \tag{1.2}$$

under the assumption that such a solution exists, or equivalently, $S_{F+G} \neq \varnothing$.

In the particular instance when $G \equiv 0$, Problem 1.2 becomes (1.1) with $H = F$, which can be solved by the methods proposed in [16, 17, 20, 25, 28]. These methods are mostly inspired from the proximal fixed point algorithm [23, 34]. The method proposed in [33] can be applied to this case when $F\colon (x,y) \mapsto \langle Bx \mid y - x\rangle$ and B is maximally monotone. On the other hand, when $G\colon (x,y) \mapsto \langle Bx \mid y - x\rangle$, where $B\colon \mathcal{H} \to \mathcal{H}$ is a cocoercive operator, weakly convergent splitting methods for solving Problem 1.2 are proposed in [11, 24]. Several methods for solving Problem 1.2 in particular instances of the bifunction G can be found in [6, 8, 9, 21, 30, 31, 32, 37] and the references therein. In the general case, sequential and parallel splitting methods are proposed in [26] with guaranteed ergodic convergence. A disadvantage of these methods is the involvement of vanishing parameters that leads to numerical instabilities, which make them of limited use in applications. The purpose of this paper is to address the general case by providing a non-ergodic weakly convergent algorithm which solves Problem 1.2. The proposed method is a consequence of the Douglas-Rachford splitting method [22, 36] applied to an auxiliary monotone inclusion involving an appropriate choice of maximally monotone operators. This choice of monotone operators allows us to deduce interesting relations between monotone equilibrium problems and monotone inclusions in Hilbert spaces. Some of these relations are deduced from related results in Banach spaces [2, 35].

The paper is organized as follows. In Section 2, we define an auxiliary monotone inclusion which is equivalent to Problem 1.2 under a suitable qualification condition, and some relations between monotone inclusions and equilibrium problems are examined. In Section 3, we propose a variant of the Douglas–Rachford splitting studied in [3, 36] and we derive our method whose iterates converge weakly to a solution of Problem 1.2. We start with some notation and useful properties.

Notation and preliminaries

Throughout this paper, $\mathcal{H}$ denotes a real Hilbert space, $\langle \cdot \mid \cdot \rangle$ denotes its inner product, and $\|\cdot\|$ denotes its induced norm. Let $\mathcal{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator. Then $\operatorname{dom}\mathcal{A} = \big\{x \in \mathcal{H} \;\big|\; \mathcal{A}x \neq \varnothing\big\}$ is the domain of $\mathcal{A}$ and $\operatorname{gra}\mathcal{A} = \big\{(x,u) \in \mathcal{H}\times\mathcal{H} \;\big|\; u \in \mathcal{A}x\big\}$ is its graph. The operator $\mathcal{A}$ is monotone if

$$(\forall (x,u) \in \operatorname{gra}\mathcal{A})(\forall (y,v) \in \operatorname{gra}\mathcal{A}) \quad \langle x-y \mid u-v\rangle \geq 0, \tag{1.3}$$

and it is called maximally monotone if its graph is not properly contained in the graph of any other monotone operator in $\mathcal{H}$. In this case, the resolvent of $\mathcal{A}$, $\mathcal{J}_{\mathcal{A}} = (\operatorname{Id} + \mathcal{A})^{-1}$, is well defined, single valued, and $\operatorname{dom}\mathcal{J}_{\mathcal{A}} = \mathcal{H}$. The reflection operator $\mathcal{R}_{\mathcal{A}} = 2\mathcal{J}_{\mathcal{A}} - \operatorname{Id}$ is nonexpansive.

For a single-valued operator $T\colon \operatorname{dom}T \subset \mathcal{H} \to \mathcal{H}$, the set of fixed points is

$$\operatorname{Fix} T = \big\{x \in \mathcal{H} \;\big|\; x = Tx\big\}. \tag{1.4}$$

We say that T is nonexpansive if

$$(\forall x \in \operatorname{dom}T)(\forall y \in \operatorname{dom}T) \quad \|Tx - Ty\| \leq \|x-y\| \tag{1.5}$$

and that T is firmly nonexpansive if

$$(\forall x \in \operatorname{dom}T)(\forall y \in \operatorname{dom}T) \quad \|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2. \tag{1.6}$$

Lemma 1.3 (*cf.* [10, Lemma 5.1]) *Let $T\colon \operatorname{dom}T = \mathcal{H} \to \mathcal{H}$ be a nonexpansive operator such that $\operatorname{Fix} T \neq \varnothing$. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $]0,1[$ and $(c_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{n\in\mathbb{N}} \mu_n(1-\mu_n) = +\infty$ and $\sum_{n\in\mathbb{N}} \mu_n\|c_n\| < +\infty$. Let $x_0 \in \mathcal{H}$ and set*

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \mu_n(Tx_n + c_n - x_n). \tag{1.7}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to $x \in \operatorname{Fix} T$ and $(x_n - Tx_n)_{n\in\mathbb{N}}$ converges strongly to 0.

Now let $F\colon C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 1.1. The resolvent of F is the operator

$$J_F\colon \mathcal{H} \to 2^C\colon x \mapsto \big\{z \in C \;\big|\; (\forall y \in C) \quad F(z,y) + \langle z - x \mid y - z\rangle \geq 0\big\}, \tag{1.8}$$

which is single valued and firmly nonexpansive [11, Lemma 2.12], and the reflection operator

$$R_F\colon \mathcal{H} \to \mathcal{H}\colon x \mapsto 2J_F x - x \tag{1.9}$$

is nonexpansive.

Let $C \subset \mathcal{H}$ be nonempty, closed, and convex. We say that 0 lies in the strong relative interior of C, in symbol, $0 \in \operatorname{sri} C$, if $\bigcup_{\lambda>0} \lambda C = \overline{\operatorname{span}}\, C$. The normal cone of C is the maximally monotone operator

$$\mathcal{N}_C\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \;\big|\; (\forall y \in C) \quad \langle y - x \mid u\rangle \le 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{1.10}$$

We denote by $\Gamma_0(\mathcal{H})$ the family of lower semicontinuous convex functions f from $\mathcal{H}$ to $\left]-\infty, +\infty\right]$ which are proper in the sense that $\operatorname{dom} f = \big\{x \in \mathcal{H} \;\big|\; f(x) < +\infty\big\}$ is nonempty. The subdifferential of $f \in \Gamma_0(\mathcal{H})$ is the maximally monotone operator $\partial f\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \big\{u \in \mathcal{H} \;\big|\; (\forall y \in \mathcal{H})\ \langle y - x \mid u\rangle + f(x) \le f(y)\big\}$. For background on convex analysis, monotone operator theory, and equilibrium problems, the reader is referred to [3, 5, 11].

2. Monotone inclusions and equilibrium problems

The basis of the method proposed in this paper for solving Problem 1.2 is that it can be formulated as finding a zero of the sum of two appropriate maximally monotone operators. In this section, we define this auxiliary monotone inclusion and, additionally, we study a class of monotone inclusions which can be formulated as an equilibrium problem.

2.1. Monotone inclusion associated to equilibrium problems

We first recall the maximal monotone operator associated to problem (1.1) and some related properties. The following result can be deduced from [2, Theorem 3.5] and [35, Proposition 4.2], which have been proved in Banach spaces.

Proposition 2.1 *Let $F\colon C \times C \to \mathbb{R}$ be such that Assumption 1.1 holds and set*

$$\mathcal{A}_F\colon \mathcal{H} \to 2^{\mathcal{H}}\colon x \mapsto \begin{cases} \big\{u \in \mathcal{H} \;\big|\; (\forall y \in C) \quad F(x,y) + \langle x - y \mid u\rangle \ge 0\big\}, & \text{if } x \in C;\\ \varnothing, & \text{otherwise.}\end{cases} \tag{2.1}$$

Then the following hold:

(i) *$\mathcal{A}_F$ is maximally monotone.*

(ii) *$S_F = \operatorname{zer} \mathcal{A}_F$.*

(iii) *For every* $\gamma \in \left]0,+\infty\right[$, $\mathcal{J}_{\gamma\mathcal{A}_F} = J_{\gamma F}$.

The following proposition allows us to formulate Problem 1.2 as an auxiliary monotone inclusion involving two maximally monotone operators obtained from Proposition 2.1.

Theorem 2.2 *Let* C, F, *and* G *be as in Problem 1.2. Then the following hold.*

(i) $\operatorname{zer}(\mathcal{A}_F + \mathcal{A}_G) \subset S_{F+G}$.

(ii) *Suppose that* $\operatorname{span}(C - C)$ *is closed. Then,* $\operatorname{zer}(\mathcal{A}_F + \mathcal{A}_G) = S_{F+G}$.

Proof. (i). Let $x \in \operatorname{zer}(\mathcal{A}_F + \mathcal{A}_G)$. Thus, $x \in C$ and there exists $u \in \mathcal{A}_F x \cap -\mathcal{A}_G x$, which yield, by (2.1),

$$\begin{cases} (\forall y \in C) \quad F(x,y) + \langle x - y \mid u \rangle \geq 0 \\ (\forall y \in C) \quad G(x,y) + \langle x - y \mid -u \rangle \geq 0. \end{cases} \tag{2.2}$$

Hence, by adding both inequalities we obtain

$$(\forall y \in C) \quad F(x,y) + G(x,y) \geq 0 \tag{2.3}$$

and, therefore, $x \in S_{F+G}$.

(ii). Let $x \in S_{F+G}$ and define

$$\begin{cases} f\colon \mathcal{H} \to \left]-\infty,+\infty\right] : y \mapsto \begin{cases} F(x,y), & \text{if } y \in C; \\ +\infty, & \text{otherwise;} \end{cases} \\ \\ g\colon \mathcal{H} \to \left]-\infty,+\infty\right] : y \mapsto \begin{cases} G(x,y), & \text{if } y \in C; \\ +\infty, & \text{otherwise.} \end{cases} \end{cases} \tag{2.4}$$

Assumption 1.1 asserts that f and g are in $\Gamma_0(\mathcal{H})$, $\operatorname{dom} f = \operatorname{dom} g = C \neq \varnothing$, and since $x \in S_{F+G}$, (1.2) yields $f + g \geq 0$. Hence, it follows from Assumption 1.1(i) and (2.4) that

$$\min_{y \in \mathcal{H}} \big(f(y) + g(y) \big) = f(x) + g(x) = 0. \tag{2.5}$$

Thus, Fermat's rule [3, Theorem 16.2] yields $0 \in \partial(f+g)(x)$. Since $\operatorname{span}(C - C)$ is closed, we have $0 \in \operatorname{sri}(C - C) = \operatorname{sri}(\operatorname{dom} f - \operatorname{dom} g)$. Therefore, it follows from [3, Corollary 16.38] that $0 \in \partial f(x) + \partial g(x)$ which implies that there exists $u_0 \in \mathcal{H}$ such that $u_0 \in \partial f(x)$ and $-u_0 \in \partial g(x)$. This is equivalent to

$$\begin{cases} (\forall y \in \mathcal{H}) \quad f(x) + \langle y - x \mid u_0 \rangle \leq f(y) \\ (\forall y \in \mathcal{H}) \quad g(x) + \langle y - x \mid -u_0 \rangle \leq g(y). \end{cases} \tag{2.6}$$

Since Assumption 1.1(i) and (2.4) yield $f(x) = g(x) = 0$, we have that (2.6) is equivalent to

$$\begin{cases} (\forall y \in C) \quad F(x,y) + \langle x - y \mid u_0 \rangle \geq 0 \\ (\forall y \in C) \quad G(x,y) + \langle x - y \mid -u_0 \rangle \geq 0. \end{cases} \tag{2.7}$$

Hence, we conclude from (2.1) that $u_0 \in \mathcal{A}_F x \cap -\mathcal{A}_G x$, which yields $x \in \operatorname{zer}(\mathcal{A}_F + \mathcal{A}_G)$. □

2.2. Equilibrium problems associated to monotone inclusions

We formulate some monotone inclusions as equilibrium problems by defining a bifunction associated to a class of maximally monotone operators. In the following proposition we present this bifunction and its properties.

Proposition 2.3 (*cf.* [11, Lemma 2.15]) *Let $\mathcal{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and suppose that $C \subset \operatorname{int}\operatorname{dom}\mathcal{A}$. Set*

$$F_{\mathcal{A}}\colon C \times C \to \mathbb{R}\colon (x,y) \mapsto \max_{u\in\mathcal{A}x} \langle y - x \mid u\rangle. \tag{2.8}$$

Then the following hold:

(i) *$F_{\mathcal{A}}$ satisfy Assumption 1.1.*
(ii) *$J_{F_{\mathcal{A}}} = \mathcal{J}_{\mathcal{A}+\mathcal{N}_C}$.*

Remark 2.4 Note that the condition $C \subset \operatorname{int}\operatorname{dom}\mathcal{A}$ allows us to take the maximum in (2.8) instead of the supremum. This is a consequence of the weakly compactness of the sets $(\mathcal{A}x)_{x\in C}$ (see [11, Lemma 2.15] for details).

Proposition 2.5 *Let $\mathcal{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator and suppose that $C \subset \operatorname{int}\operatorname{dom}\mathcal{A}$. Then $\operatorname{zer}(\mathcal{A}+\mathcal{N}_C) = S_{F_{\mathcal{A}}}$.*

Proof. Indeed, it follows from [3, Proposition 23.38], Proposition 2.3, and [11, Lemma 2.15(i)] that

$$\operatorname{zer}(\mathcal{A}+\mathcal{N}_C) = \operatorname{Fix}(\mathcal{J}_{\mathcal{A}+\mathcal{N}_C}) = \operatorname{Fix}(J_{F_{\mathcal{A}}}) = S_{F_{\mathcal{A}}}, \tag{2.9}$$

which yields the result. □

Remark 2.6

(i) Note that, in the particular case when $\operatorname{dom}\mathcal{A} = \operatorname{int}\operatorname{dom}\mathcal{A} = C = \mathcal{H}$, Proposition 2.5 asserts that $\operatorname{zer}\mathcal{A} = S_{F_{\mathcal{A}}}$, which is a well known result (e.g., see [19, Section 2.1.3]).
(ii) In Banach spaces, the case when $C = \operatorname{dom}\mathcal{A} \subset \mathcal{H}$ is studied in [2, Theorem 3.8].

The following propositions provide a relation between the operators defined in Propositions 2.1 and 2.3.

Proposition 2.7 *Let $\mathcal{B}\colon \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone and suppose that $C \subset \operatorname{int}\operatorname{dom}\mathcal{B}$. Then, $\mathcal{A}_{F_{\mathcal{B}}} = \mathcal{B} + \mathcal{N}_C$.*

Proof. Let $(x,u) \in \mathcal{H}^2$. It follows from (2.8) and [5, Lemma 1] (see also [11, Lemma 2.14]) that

$$\begin{aligned}
u \in \mathcal{A}_{F_{\mathcal{B}}} x \quad &\Leftrightarrow \quad x \in C \quad \text{and} \quad (\forall y \in C) \quad F_{\mathcal{B}}(x,y) + \langle x - y \mid u \rangle \geq 0 \\
&\Leftrightarrow \quad x \in C \quad \text{and} \quad (\forall y \in C) \quad \max_{v \in \mathcal{B}x} \langle y - x \mid v \rangle + \langle x - y \mid u \rangle \geq 0 \qquad (2.10) \\
&\Leftrightarrow \quad x \in C \quad \text{and} \quad (\forall y \in C) \quad \max_{v \in \mathcal{B}x} \langle y - x \mid v - u \rangle \geq 0 \qquad (2.11) \\
&\Leftrightarrow \quad x \in C \quad \text{and} \quad (\exists\, v \in \mathcal{B}x)(\forall y \in C) \quad \langle y - x \mid v - u \rangle \geq 0 \qquad (2.12) \\
&\Leftrightarrow \quad u \in \mathcal{B}x + \mathcal{N}_C x, \qquad (2.13)
\end{aligned}$$

which yields the result. ☐

Proposition 2.8 *Let G be such that Assumption 1.1 holds, and suppose that $C = \operatorname{dom} \mathcal{A}_G = \mathcal{H}$. Then, $F_{\mathcal{A}_G} \leq G$.*

Proof. Let $(x,y) \in C \times C$ and let $u \in \mathcal{A}_G x$. It follows from (2.1) that $G(x,y) + \langle x - y \mid u \rangle \geq 0$, which yields

$$(\forall u \in \mathcal{A}_G x) \quad \langle y - x \mid u \rangle \leq G(x,y). \qquad (2.14)$$

Since $C = \operatorname{int} \operatorname{dom} \mathcal{A}_G = \mathcal{H}$, the result follows by taking the maximum in the left side of the inequality. ☐

Remark 2.9

(i) Note that the equality in Proposition 2.8 does not hold in general. Indeed, let $\mathcal{H} = \mathbb{R}$, $C = \mathcal{H}$, and $G\colon (x,y) \mapsto y^2 - x^2$. It follows from [11, Lemma 2.15(v)] that G satisfy Assumption 1.1. We have $u \in \mathcal{A}_G x \Leftrightarrow (\forall y \in \mathcal{H}) \quad y^2 - x^2 + \langle x - y \mid u \rangle \geq 0 \Leftrightarrow u = 2x$ and, hence, for every $(x,y) \in \mathcal{H} \times \mathcal{H}$, $F_{\mathcal{A}_G}(x,y) = (y-x)2x = 2xy - 2x^2$. In particular, for every $y \in \mathbb{R} \setminus \{0\}$, $F_{\mathcal{A}_G}(0,y) = 0 < y^2 = G(0,y)$.

(ii) In the general case when $C = \operatorname{dom} \mathcal{A} \subset \mathcal{H}$, necessary and sufficient conditions for the equality in Proposition 2.8 are provided in [2, Theorem 4.5].

3. Algorithm and convergence

Theorem 2.2(ii) characterizes the solutions to Problem 1.2 as the zeros of the sum of two maximally monotone operators. Our algorithm is derived from the Douglas-Rachford splitting method for solving this auxiliary monotone inclusion. This algorithm was first proposed in [13] in finite dimensional spaces when the operators are linear and the generalization to general maximally monotone operators in Hilbert spaces was first developed in [22]. Other versions involving computational errors of the resolvents can be found in [10, 14]. The convergence of these methods needs the maximal monotonicity of the sum of the operators involved, which is not evident to verify [3, Section 24.1]. Furthermore, the iterates in these cases do not converge to a solution but to a point from which we can calculate a solution. These problems were overcame in [36] and, later, in [3, Theorem 25.6], where the convergence of the sequences generated by the proposed methods to a zero of the sum of two set-valued operators is guaranteed by only assuming the maximal monotonicity of each operator. However, in [36] the errors considered do not come from inaccuracies on the computation of the resolvent but only from imprecisions in a monotone inclusion, which sometimes could be not manipulable. On the other hand, in [3, Theorem 25.6] the method includes an additional relaxation step but it does not consider inaccuracies in its implementation.

We present a variant of the methods presented in [36] and [3, Theorem 25.6], which has interest in its own right. The same convergence results are obtained by considering a relaxation step as in [14, 15] and errors in the computation of the resolvents as in [10, 14].

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two maximally monotone operators from $\mathcal{H}$ to $2^{\mathcal{H}}$ such that $\operatorname{zer}(\mathcal{A}+\mathcal{B}) \neq \varnothing$. Let $\gamma \in \left]0,+\infty\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0,2\right[$, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $b_n \rightharpoonup 0$,*

$$\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty, \quad \text{and} \quad \sum_{n\in\mathbb{N}} \lambda_n(\|a_n\| + \|b_n\|) < +\infty. \tag{3.1}$$

Let $x_0 \in \mathcal{H}$ and set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = \mathcal{J}_{\gamma\mathcal{B}} x_n + b_n \\ z_n = \mathcal{J}_{\gamma\mathcal{A}}(2y_n - x_n) + a_n \\ x_{n+1} = x_n + \lambda_n(z_n - y_n). \end{cases} \tag{3.2}$$

Then there exists $x \in \operatorname{Fix}(\mathcal{R}_{\gamma\mathcal{A}}\mathcal{R}_{\gamma\mathcal{B}})$ such that the following hold:

(i) $\mathcal{J}_{\gamma\mathcal{B}} x \in \operatorname{zer}(\mathcal{A}+\mathcal{B})$.
(ii) $(\mathcal{R}_{\gamma\mathcal{A}}(\mathcal{R}_{\gamma\mathcal{B}} x_n) - x_n)_{n\in\mathbb{N}}$ *converges strongly to* 0.
(iii) $(x_n)_{n\in\mathbb{N}}$ *converges weakly to* x.
(iv) $(y_n)_{n\in\mathbb{N}}$ *converges weakly to* $\mathcal{J}_{\gamma\mathcal{B}} x$.

Proof. Denote $\mathcal{T} = \mathcal{R}_{\gamma\mathcal{A}}\mathcal{R}_{\gamma\mathcal{B}}$. Since $\mathcal{R}_{\gamma\mathcal{A}}$ and $\mathcal{R}_{\gamma\mathcal{B}}$ are nonexpansive operators, $\mathcal{T}$ is nonexpansive as well. Moreover, since [3, Proposition 25.1(ii)] states that $\mathcal{J}_{\gamma\mathcal{B}}(\operatorname{Fix}\mathcal{T}) = \operatorname{zer}(\mathcal{A}+\mathcal{B})$, we deduce that $\operatorname{Fix}\mathcal{T} \neq \varnothing$. Note that (3.2) can be rewritten as

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = x_n + \mu_n(\mathcal{T}x_n + c_n - x_n), \tag{3.3}$$

where, for every $n \in \mathbb{N}$,

$$\mu_n = \frac{\lambda_n}{2} \quad \text{and} \quad c_n = 2\big(\mathcal{J}_{\gamma\mathcal{A}}\big(2(\mathcal{J}_{\gamma\mathcal{B}} x_n + b_n) - x_n\big) + a_n - \mathcal{J}_{\gamma\mathcal{A}}(2\mathcal{J}_{\gamma\mathcal{B}} x_n - x_n) - b_n\big). \tag{3.4}$$

Hence, it follows from the nonexpansivity of $\mathcal{J}_{\gamma\mathcal{A}}$ that, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\|c_n\| &= 2\|\mathcal{J}_{\gamma\mathcal{A}}\big(2(\mathcal{J}_{\gamma\mathcal{B}}x_n + b_n) - x_n\big) + a_n - \mathcal{J}_{\gamma\mathcal{A}}(2\mathcal{J}_{\gamma\mathcal{B}}x_n - x_n) - b_n\| \\
&\leq 2\|\mathcal{J}_{\gamma\mathcal{A}}\big(2(\mathcal{J}_{\gamma\mathcal{B}}x_n + b_n) - x_n\big) - \mathcal{J}_{\gamma\mathcal{A}}(2\mathcal{J}_{\gamma\mathcal{B}}x_n - x_n)\| + 2\|a_n\| + 2\|b_n\| \\
&\leq 2\|2(\mathcal{J}_{\gamma\mathcal{B}}x_n + b_n) - x_n - (2\mathcal{J}_{\gamma\mathcal{B}}x_n - x_n)\| + 2\|a_n\| + 2\|b_n\| \\
&= 2(\|a_n\| + 3\|b_n\|)
\end{aligned} \tag{3.5}$$

and, therefore, from (3.1) and (3.4) we obtain

$$\sum_{n\in\mathbb{N}} \mu_n\|c_n\| \leq \sum_{n\in\mathbb{N}} \lambda_n(\|a_n\| + 3\|b_n\|) \leq 3\sum_{n\in\mathbb{N}} \lambda_n(\|a_n\| + \|b_n\|) < +\infty. \tag{3.6}$$

Moreover, since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is in $]0,2[$, it follows from (3.4) that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $]0,1[$ and, from (3.1) we obtain

$$\sum_{n\in\mathbb{N}} \mu_n(1-\mu_n) = \frac{1}{4}\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty. \tag{3.7}$$

(i). This follows from [3, Proposition 25.1(ii)].

(ii) and (iii). These follow from Lemma 1.3.

(iv). From the nonexpansivity of $\mathcal{J}_{\gamma\mathcal{B}}$ we obtain

$$\|y_n - y_0\| \leq \|\mathcal{J}_{\gamma\mathcal{B}}x_n - \mathcal{J}_{\gamma\mathcal{B}}x_0\| + \|b_n - b_0\| \leq \|x_n - x_0\| + \|b_n - b_0\|. \tag{3.8}$$

It follows from (iii) and $b_n \rightharpoonup 0$ that $(x_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded, respectively. Hence, (3.8) implies that $(y_n)_{n\in\mathbb{N}}$ is bounded as well. Let $y \in \mathcal{H}$ be a weak sequential cluster point of $(y_n)_{n\in\mathbb{N}}$, say $y_{k_n} \rightharpoonup y$, and set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} \widetilde{y}_n = \mathcal{J}_{\gamma\mathcal{B}}x_n \\ \widetilde{z}_n = \mathcal{J}_{\gamma\mathcal{A}}(2\widetilde{y}_n - x_n) \\ \widetilde{u}_n = 2\widetilde{y}_n - x_n - \widetilde{z}_n \\ \widetilde{v}_n = x_n - \widetilde{y}_n. \end{cases} \tag{3.9}$$

It follows from (3.2) that

$$(\forall n \in \mathbb{N}) \quad \begin{cases} (\widetilde{z}_n, \widetilde{u}_n) \in \operatorname{gra}\gamma\mathcal{A} \\ (\widetilde{y}_n, \widetilde{v}_n) \in \operatorname{gra}\gamma\mathcal{B} \\ \widetilde{u}_n + \widetilde{v}_n = \widetilde{y}_n - \widetilde{z}_n. \end{cases} \tag{3.10}$$

For every $n \in \mathbb{N}$, we obtain from (3.9)

$$\begin{aligned}\|\widetilde{z}_{k_n} - \widetilde{y}_{k_n}\| &= \|\mathcal{J}_{\gamma\mathcal{A}}(2\mathcal{J}_{\gamma\mathcal{B}}x_{k_n} - x_{k_n}) - \mathcal{J}_{\gamma\mathcal{B}}x_{k_n}\| \\ &= \frac{1}{2}\|2\mathcal{J}_{\gamma\mathcal{A}}(2\mathcal{J}_{\gamma\mathcal{B}}x_{k_n} - x_{k_n}) - (2\mathcal{J}_{\gamma\mathcal{B}}x_{k_n} - x_{k_n}) - x_{k_n}\| \\ &= \frac{1}{2}\|\mathcal{R}_{\gamma\mathcal{A}}(\mathcal{R}_{\gamma\mathcal{B}}x_{k_n}) - x_{k_n}\|. \end{aligned} \tag{3.11}$$

Hence, (ii) yields $\widetilde{z}_{k_n} - \widetilde{y}_{k_n} \to 0$, and, therefore, from (3.10) we obtain that $\widetilde{u}_{k_n} + \widetilde{v}_{k_n} \to 0$. Moreover, it follows from $b_{k_n} \rightharpoonup 0$, $y_{k_n} \rightharpoonup y$, and (3.2) that $\widetilde{y}_{k_n} \rightharpoonup y$, and, hence, $\widetilde{z}_{k_n} \rightharpoonup y$. Thus, from (iii) and (3.9), we obtain $\widetilde{u}_{k_n} \rightharpoonup y - x$ and $\widetilde{v}_{k_n} \rightharpoonup x - y$. Altogether, from [3, Corollary 25.5] we deduce that $y \in \operatorname{zer}(\gamma\mathcal{A} + \gamma\mathcal{B}) = \operatorname{zer}(\mathcal{A} + \mathcal{B})$, $(y, y - x) \in \operatorname{gra}\gamma\mathcal{A}$, and $(y, x - y) \in \operatorname{gra}\gamma\mathcal{B}$. Hence, $y = \mathcal{J}_{\gamma\mathcal{B}}x$ and $y \in \operatorname{dom}\mathcal{A}$. Therefore, we conclude that $\mathcal{J}_{\gamma\mathcal{B}}x$ is the unique weak sequential cluster point of $(y_n)_{n\in\mathbb{N}}$ and then $y_n \rightharpoonup \mathcal{J}_{\gamma\mathcal{B}}x$. ☐

Now we present our method for solving Problem 1.2, which is an application of Theorem 3.1 to the auxiliary monotone inclusion obtained in Theorem 2.2.

Theorem 3.2 *Let C, F, and G be as in Problem 1.2 and suppose that $\operatorname{span}(C - C)$ is closed. Let $\gamma \in \,]0, +\infty[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $]0, 2[$, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathcal{H}$ such that $b_n \rightharpoonup 0$,*

$$\sum_{n\in\mathbb{N}} \lambda_n(2 - \lambda_n) = +\infty, \quad \text{and} \quad \sum_{n\in\mathbb{N}} \lambda_n(\|a_n\| + \|b_n\|) < +\infty. \tag{3.12}$$

Let $x_0 \in \mathcal{H}$ and set

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = J_{\gamma G}x_n + b_n \\ z_n = J_{\gamma F}(2y_n - x_n) + a_n \\ x_{n+1} = x_n + \lambda_n(z_n - y_n). \end{cases} \tag{3.13}$$

Then there exists $x \in \operatorname{Fix}(R_{\gamma F}R_{\gamma G})$ such that the following hold:

(i) *$J_{\gamma G}x \in S_{F+G}$.*
(ii) *$(R_{\gamma F}(R_{\gamma G}x_n) - x_n)_{n\in\mathbb{N}}$ converges strongly to 0.*
(iii) *$(x_n)_{n\in\mathbb{N}}$ converges weakly to x.*
(iv) *$(y_n)_{n\in\mathbb{N}}$ converges weakly to $J_{\gamma G}x$.*

Proof. Note that, from Theorem 2.2(ii), we have that

$$\operatorname{zer}(\mathcal{A}_F + \mathcal{A}_G) = S_{F+G} \neq \varnothing, \tag{3.14}$$

where $\mathcal{A}_F$ and $\mathcal{A}_G$ are defined in (2.1) and maximally monotone by Proposition 2.1(i). In addition, it follows from Proposition 2.1(iii) that (3.13) can be written equivalently as (3.2) with $\mathcal{A} = \mathcal{A}_F$ and $\mathcal{B} = \mathcal{A}_G$. Hence, the results are derived from Theorem 3.1, Proposition 2.1, and Theorem 2.2. ☐

Remark 3.3 Note that the closeness of $\mathrm{span}(C-C)$ and Theorem 2.2(ii) yields (3.14), which allows us to apply Theorem 3.1 for obtaining our result. However, it is well known that this qualification condition does not always hold in infinite dimensional spaces. In such cases, it follows from Theorem 2.2(i) that Theorem 3.2 still holds if $\mathrm{zer}(\mathcal{A}_F + \mathcal{A}_G) \neq \varnothing$. Conditions for assuring existence of solutions to monotone inclusions can be found in [7, Proposition 3.2] and [3].

Finally, let us show an application of Theorem 3.2 for solving mixed equilibrium problems. Let $f \in \Gamma_0(\mathcal{H})$. For every $x \in \mathcal{H}$, $\mathrm{prox}_f\, x$ is the unique minimizer of the strongly convex function $y \mapsto f(y) + \|y - x\|^2/2$. The operator $\mathrm{prox}_f \colon \mathcal{H} \to \mathcal{H}$ thus defined is called the proximity operator.

Example 3.4 In Problem 1.2, suppose that $G\colon (x, y) \mapsto f(y) - f(x)$, where $f \in \Gamma_0(\mathcal{H})$ is such that $C \subset \mathrm{dom}\, f$. Then Problem 1.2 becomes

$$\text{find} \quad x \in C \quad \text{such that} \quad (\forall y \in C) \quad F(x, y) + f(y) \geq f(x), \tag{3.15}$$

which is known as a mixed equilibrium problem. This problem arises in several applied problems and it can be solved by using some methods developed in [8, 32, 30, 37]. However, all this methods consider implicit steps involving simultaneously F and f, which is not easy to compute in general. On the other hand, it follows from [11, Lemma 2.15(v)] that (3.13) becomes

$$(\forall n \in \mathbb{N}) \quad \begin{cases} y_n = \mathrm{prox}_{\iota_C + \gamma f}\, x_n + b_n \\ z_n = J_{\gamma F}(2y_n - x_n) + a_n \\ x_{n+1} = x_n + \lambda_n(z_n - y_n), \end{cases} \tag{3.16}$$

which computes separately the resolvent of F and the proximity operator of f. If $\mathrm{span}(C-C)$ is closed, Theorem 3.2 ensures the weak convergence of the iterates of this method to a solution to (3.15). Examples of computable proximity operators and resolvents of bifunctions can be found in [12] and [11], respectively.

4. Acknowledgement

I thank Professor Patrick L. Combettes for bringing this problem to my attention and for helpful discussions. In addition, I would like to thank the anonymous reviewers for their comments that help improve the manuscript.

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